SOME 6 DIMENSIONAL HAMILTONIAN S^1-MANIFOLDS

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ABSTRACT. In an earlier paper we explained how to convert the problem of symplectically embedding one 4-dimensional ellipsoid into another into the problem of embedding a certain set of disjoint balls into $\mathbb{C}P^2$ by using a new way to desingularize orbifold blow ups Z of the weighted projective space $\mathbb{C}P^2_{1,m,n}$. We now use a related method to construct symplectomorphisms of these spaces Z. This allows us to construct some well known Fano 3-folds (including the Mukai–Umemura 3-fold) in purely symplectic terms using a classification by Tolman of a particular class of Hamiltonian S^1-manifolds. We also show that (modulo scaling) these manifolds are uniquely determined by their fixed point data up to equivariant symplectomorphism. As part of this argument we show that the symplectomorphism group of a certain weighted blow up of a weighted projective plane is connected.

CONTENTS

1. Introduction

1.1. **Statement of results.** In [29], Tolman considers the problem of classifying all symplectic 6-manifolds (M, Ω) with a Hamiltonian S^1 action in the case when $H^2(M; \mathbb{Z})$ has rank 1. She proved that under these assumptions $H^*(M; \mathbb{Z})$ is additively isomorphic to $H^*(\mathbb{C}P^3; \mathbb{Z}) = \mathbb{Z}$ and that there are four possibilities for the number $\ell := 6 - c_1(\beta)$ where β is the generator of $H_2(M)$ with $\omega(\beta) > 0$. The two standard cases are $M_2 = \mathbb{C}P^3$ and $M_3 = \widetilde{G}_{\mathbb{R}}(2, 5)$, the Grassmannian of oriented 2-planes in $\mathbb{R}^5$ (known to complex geometers as the quadric surface in $\mathbb{C}P^4$).

However there are two other possibilities, with $\ell = 4$ or 5. In the latter two cases Tolman showed that S^1 must act with precisely 4 fixed points $x_k, 1 \le k \le 4$, that have index $8 - 2k$ and isotropy weights w_k, where

$$w_1 = (-1, -2, -3), \quad w_2 = (1, -1, -\ell), \quad w_3 = (1, \ell, -1), \quad w_4 = (1, 2, 3).$$

Moreover, the generating Hamiltonian H (or moment map) can be chosen[1] to have critical levels

$$H(x_1) = 6, \quad H(x_2) = \ell, \quad H(x_3) = -\ell, \quad H(x_4) = -6,$$

and the integral cohomology ring $H^*(M_\ell; \mathbb{Z})$ must have the following form: if $x \in H^2$ and $y \in H^4$ are generators such that $x(\beta) = 1$ and xy generates H^6, then

$$x^2 = 5y \text{ when } \ell = 4, \text{ and } x^2 = 22y \text{ when } \ell = 5.$$

Tolman showed that this data satisfies many consistency checks. However she left open the question as to whether manifolds with $\ell = 4, 5$ actually exist.

It turns out that these manifolds are well known to complex geometers. Any Hamiltonian S^1 manifold contains 2-spheres on which ω is positive; take the S^1-orbit of any g_J-gradient flow line of the moment map H, where $g_J := \omega(\cdot, J\cdot)$ is defined using a compatible almost complex structure J. Hence, if complex, these manifolds would be Fano 3-folds with $b_2 = 1$ and $b_3 = 0$. Such manifolds are classified (see [8, Ch 12]). There are precisely four families, corresponding to the four cases $\ell = 2, 3, 4, 5$ discussed above. Rather than using the number ℓ, algebraic geometers distinguish them by their index $r := c_1(\beta) = 6 - \ell$. When $r = 4$ one has $\mathbb{C}P^3$ and when $r = 3$ the quadric. There is a unique complex manifold V_5 (also sometimes called B_5) with index 2 which is rigid (i.e. its complex structure does not deform); it supports a nontrivial action of $SL(2, \mathbb{C})$. In contrast, when $r = 1$ there is a family V_{22} of manifolds. As shown by Prokhorov [26], there is a unique member of this family V_{22}^s with a nontrivial $SL(2, \mathbb{C})$ action, another unique member V_{22}^a with an action of $\mathbb{C}$ and a family V_{22}^m depending on one rational parameter with an action of $\mathbb{C}^*$. The manifold V_{22}^s was first constructed by Mukai–Umemura [22] and is of particular interest to geometers because of its Kähler–Einstein metrics; cf. Donaldson [4] for example.

In this paper we construct the manifolds $M_4 = V_5$ and $M_5 = V_{22}$ in purely symplectic terms. We also show that they admit complex structures that are invariant under an

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[1]There are two choices here. The first is to choose the additive constant for H so that it is symmetric about 0, and the second is to scale the symplectic form so that it equals $c_1(M)$.

S^1 action and hence under a $\mathbb{C}^*$ action. Because they are Fano, they also have S^1-invariant Kähler structures induced by the embeddings into projective space provided by sections of high enough powers of the anticanonical bundle. Our method does not exhibit the $SO(3)$ action (but see Remark 4.6 and [4, §5.2]).

Theorem 1.1. (i) *When $\ell = 4, 5$, there are Hamiltonian S^1 manifolds (M_ℓ, Ω) with the properties described above. Modulo scaling, they are unique up to S^1-equivariant symplectomorphism.*

(ii) *Moreover these manifolds may be given an S^1-invariant complex structure. This is unique when $\ell = 4$, and depends on a rational parameter when $\ell = 5$.*

The only new statement above is the uniqueness part of (i). Its proof takes the approach proposed by Gonzalez [7] and relies on Theorem 2.16 which states that the reduced spaces are "rigid", i.e. that their symplectic structures are unique in a fairly strong sense. The construction of the complex structures in (ii) is rather different from those in the original papers (cf. [22]), and provides a new perspective on the discussion of the $SO(3)$ action in Donaldson [4, §5.2].

We analyze the symplectic structure of (M_ℓ, Ω) via the family of reduced spaces. As is explained in more detail below, these reduced spaces are 4-dimensional symplectic orbifolds. Rather than looking at them directly as in Chen [2], we study them via their symplectic resolution as in McDuff [17]. The resolution of the middle reduced level is the blow up X_k of $\mathbb{C}P^2$ at $k := \ell + 3$ points, and our first construction is based on the existence of certain elements of order two (the Geiser and Bertini involutions) in the plane Cremona group; cf. Remark 2.2(ii). Although our method is applied here only in a special case, in principle it could be used to construct any 6 dimensional Hamiltonian S^1-manifold with isolated fixed points once one has a consistent set of fixed point data. However, the uniqueness result uses the fact the resolution involves a relatively small number of blow ups, and may well not hold in general. Note also that the existence of complex structures on M_ℓ is established by a somewhat different argument, one relying on the existence of very special complex structures that are invariant under analogs of the above involutions: see §4.

1.2. **Sketch of proof.** We now sketch our argument in the symplectic case. In [6], Godinho analysed the change in structure of the reduced spaces of a Hamiltonian S^1-manifold when one passes through a critical point of index or coindex 2. Her work implies that if the manifolds M_ℓ exist then the regular reduced spaces $(Z_\kappa, \omega_\kappa)$ at level $\kappa \in (-6, 6)$ must be certain orbifold blow ups of weighted projective spaces. Tolman worked out precisely what these reduced spaces must look like (see Lemma 2.4 below), and pointed out that the question of whether they actually exist is equivalent to an ellipsoidal embedding problem. The latter problem was solved in [17]. It follows immediately that the sub- and super-level sets

$$\left(M_\ell^{\leq 0}, \Omega\right) := \left(H^{-1}([-6, 0]), \Omega\right), \quad \left(M_\ell^{\geq 0}, \Omega\right) := \left(H^{-1}([0, 6]), \Omega\right)$$

of $M_\ell, \ell = 4, 5$, also exist. Therefore all we need to do is glue the boundary of $\left(M_\ell^{\leq 0}, \Omega\right)$ to that of $\left(M_\ell^{\geq 0}, \Omega\right)$.

If Z_0 had no singularities, this would amount to constructing the symplectic sum of the cut symplectic manifolds (M^-, Z^-, ω^-) and (M^+, Z^+, ω^+) along the copies Z^-, Z^+ of Z_0, where M^-, for example, is obtained from $M^{\leq 0}$ by collapsing each S^1 orbit in its boundary to a point in Z^-. For this sum operation to be possible we need there to be a symplectomorphism $(Z^-, \omega^-) \to (Z^+, \omega^+)$ that reverses the sign of the Euler class of the normal bundles. In the case at hand, the boundary $(Y^-, \Omega^-) := (H^{-1}(0), \Omega) \subset M^{\leq 0}$ is the (smooth) total space of a principal S^1-orbibundle $\pi : (Y^-, \Omega^-) \to (Z^-, \omega_0^-)$ over the reduced space $(Z^-, \omega_0^-) := (Y^-/S^1, \omega^-)$, which is a symplectic orbifold whose singular set $\mathbf{p}$ consists of 3 points. It is not hard to see that the orbibundle $Y^- \to Z^-$ is determined by its restriction to $Z^- \smallsetminus \mathbf{p}$. Since the latter is a circle bundle, it is in turn determined by its Euler class $e(Y^-) \in H^2(Z^- \smallsetminus \mathbf{p}; \mathbb{Z})$. But, as we shall see in §2.2, $H^2(Z^- \smallsetminus \mathbf{p}; \mathbb{Z})$ is a free abelian group, and the restriction map $H^2(Z^-; \mathbb{Q}) \to H^2(Z^- \smallsetminus \mathbf{p}; \mathbb{Q})$ is an isomorphism. Hence the orbibundle $Y^- \to Z^-$ is determined by the unique class $e_Z(Y^-) \in H^2(Z^-; \mathbb{Q})$ that restricts to $e(Y^-)$. This leads to the following statement.

Lemma 1.2. *To construct (M_ℓ, Ω) as a Hamiltonian S^1-manifold, it suffices to find a symplectomorphism $\phi_Z : (Z^-, \omega_0^-) \to (Z^+, \omega_0^+)$ such that $\phi_Z^*(e_Z(Y^+)) = -e_Z(Y^-)$.*

Therefore the first part of the following result gives the existence statement of Theorem 1.1 (i), while the second part will imply the uniqueness statement via Lemma 3.2.

Proposition 1.3. (i) *For $\ell = 4, 5$, there is a symplectomorphism $\phi_Z : (Z^-, \omega_0^-) \to (Z^+, \omega_0^+)$ such that $\phi_Z^*(e_Z(Y^+)) = -e_Z(Y^-)$.*
(ii) *Moreover ϕ_Z is unique up to symplectic isotopy.*

To prove this we resolve Z as follows. Denote by (X_k, J_0) the complex manifold obtained by blowing up $\mathbb{C}P^2$ at k generic points, and by $L, E_i, 1 \leq i \leq k$, the classes of the line $\mathbb{C}P^1$ and the k exceptional divisors. We shall provide (X_k, J_0) with a J_0-tame symplectic form in the class

$$(1.1) \qquad\qquad [\tau] = 3a - \sum_{i=1}^{k} e_i = c_1(X_k, J_k),$$

where a, e_i are Poincaré dual to L, E_i respectively. In particular, $e_i(E_j) = -\delta_{ij}$. Note that it does not matter here how we choose J_0 or the symplectic form; by [15], any choices give forms that are deformation equivalent and hence isotopic. Further, define

$$(1.2) \qquad \begin{aligned} \chi_7 : &= \tfrac{1}{12}\big(6a - \sum_{1 \leq i \leq 3} 2e_i - \sum_{4 \leq i \leq 7} 3e_i\big) \quad \text{on } X_7, \\ \chi_8 : &= \tfrac{1}{30}\big(15a - \sum_{1 \leq i \leq 3} 5e_i - \sum_{4 \leq i \leq 8} 6e_i\big) \quad \text{on } X_8. \end{aligned}$$

We shall see in §2 that there is a complex structure $J(\neq J_0)$ on $X_{\ell+3}$ and a holomorphic blow down map $\Phi_J : (X_{\ell+3}, J) \to Z^-$ such that

$$\chi_{\ell+3} = -\Phi_J^*(e_Z(Y^-)), \quad [\tau] = \Phi_J^*([\omega_0]).$$

In fact, if one thinks of Z^- as a complex orbifold, Φ_J is just a standard resolution of its singularities; the results of [17] are needed only to understand the symplectic structure

of Z^-. Similarly, there is a holomorphic blow down map $\Phi_{J'} : (X_{\ell+3}, J') \to Z^+$ such that $\chi_{\ell+3} = \Phi_{J'}^*(e_Z(Y^+))$. These facts, together with Proposition 2.20 concerning the uniqueness of symplectic forms on $Z^\pm$, allow us to reduce the proof of Proposition 1.3 (i) to the following lemma.

Lemma 1.4. *For $k = 7, 8$, there is a diffeomorphism $\psi : X_k \to X_k$ such that $\psi^*(\chi_k) = -\chi_k$.*

This result is classical (cf. Remark 2.2), but we prove it in §2.1 for the sake of completeness. This completes the construction of (M_ℓ, Ω) as a symplectic manifold. Here the resolution X_k is for the most part considered as a complex manifold and we use the holomorphic blow down map $\Phi_J : X_k \to Z$. However, to prove uniqueness we need to understand the symplectic structure of Z much more deeply. In particular the following result is proved in §2.3.

Proposition 1.5. *For any symplectic structure on the orbifold Z the group of symplectomorphisms that act trivially on homology is connected.*

The proof uses the symplectic version of the resolution. In Lemmas 3.2 and 3.4 we also give proofs of basic uniqueness results for suitable slices $H^{-1}(a, b)$ of Hamiltonian S^1-manifolds. These lemmas are well known, but there is no convenient reference in the literature.

Remark 1.6. (i) We explain in §3.3 a similar construction for the manifolds $M_2 = \mathbb{C}P^3$ and $M_3 = \widetilde{G}_\mathbb{R}(2, 5)$. Since the S^1 action on M_2 extends to a Hamiltonian action of T^3, the reduced spaces in this case are toric, with moment polytopes given by a family of parallel slices of the 3-simplex that is illustrated in Figure 3.3.

(ii) M_4 and M_5 admit Hamiltonian $SO(3)$ actions, and it would be interesting to use the methods of River Chang [1] to understand them up to $SO(3)$-equivariant symplectomorphism. More generally, it would be interesting to understand when a Hamiltonian S^1 action extends to an $SO(3)$ action; can one give conditions on the reduced spaces that would guarantee this? The toric version of this question is understood. For example, it is shown in McDuff–Tolman [21] that a toric manifold admits a compatible $SO(3)$ action if and only if the moment polytope admits a nontrivial robust affine symmetry; cf. [21] Lemma 1.26 and Proposition 5.5.

2. Blow ups of $\mathbb{C}P^2$ and weighted projective spaces.

2.1. **Symplectomorphisms of X_k.** In this section we shall prove Lemma 1.4 in the more precise form given by Proposition 2.1 below. We begin with a general discussion of automorphisms of X_k. One difficulty in making this discussion precise is that there are serious gaps in our knowledge of the group $\mathrm{Diff}(X_k)$ of diffeomorphisms of X_k. In particular, even when $k = 0$, i.e. for $X_0 = \mathbb{C}P^2$, it is not known whether the subgroup $\mathrm{Diff}_H(X_k)$ that acts trivially on homology is connected, though the group of symplectomorphisms of $\mathbb{C}P^2$ is connected by Gromov's results.

For all k we shall denote by J_0 the complex structure on X_k obtained by identifying X_k with the blow up of $\mathbb{C}P^2$ at a particular set of k generic points. We shall assume that (X_ℓ, J_0) is a blow up of (X_k, J_0) for all $\ell > k$ and write K for its canonical class. Thus $-K = 3L - \sum_{i=1}^{k} E_i$.

We shall denote by $\mathcal{E}(X_k)$ the set of classes in $H_2(X_k)$ that can be represented by embedded -1 spheres. Thus $\mathcal{E}(X_k) = \{E \in H_2(X_k) : E^2 = -1, K \cdot E = -1\}$. When $k \leq 8$ the elements of $\mathcal{E}(X_k)$ can be listed as follows (modulo permutations of the indices)

$$(2.1) \qquad E_1, \ \ L - E_{12}; \ \ 2L - E_{1\dots5}; \ \ 3L - 2E_1 - E_{2\dots7};$$
$$4L - 2E_{123} - E_{4\dots8}; \ \ 5L - 2E_{1\dots6} - E_{78}; \ \ 6L - 3E_1 - 2E_{2\dots8}.$$

(Here we denote $\sum_{i=j}^{n} E_i =: E_{j\dots n}$. Further, elements of the last three kinds do not appear in $\mathcal{E}(X_7)$ since they involve 8 different E_i.)

Next, recall that the classical *Cremona transformation* $R_{123} : X_3 \to X_3$ is the biholomorphism that covers the birational map

$$\rho : \mathbb{C}P^2 \smallsetminus \{3 \text{ pts}\} \ \longrightarrow \ \mathbb{C}P^2 \smallsetminus \{3 \text{ pts}\}, \qquad [x : y : z] \mapsto [yz : xz : xy].$$

Thus R_{123} acts on $H_2(X_3)$ by

$$L \mapsto 2L - E_{123}, \qquad E_i \mapsto L - E_j - E_k,$$

where $\{i, j, k\} = \{1, 2, 3\}$. By Seidel [27], R_{123} is isotopic to a symplectomorphism of X_3, when this has a J_0-tame symplectic form in the class Poincaré dual to $-K = 3L - E_{123}$. Indeed, in this case R_{123} is isotopic to the Dehn twist in a Lagrangian sphere in class $L - E_{123}$.

We denote the Cremona transformation of X_k in the exceptional divisors E_i, E_j, E_ℓ by $R_{ij\ell}$. It is well defined up to isotopy, and acts on $H_2(X_k)$ by the reflection $A \mapsto A + (A \cdot B)B$ where $B := L - E_{ij\ell}$.

Denote by $\mathrm{Aut}_K(X_k)$ the group of automorphisms of the homology group $H_2(X_k; \mathbb{Z})$ that preserve the canonical class K and the intersection form. Further, denote by $\mathrm{Diff}_K(X_k)$ the group of diffeomorphisms of X_k that preserve K. A classical result of Wall [30] asserts that the natural map $\pi_0(\mathrm{Diff}_K(X_k)) \to \mathrm{Aut}_K(X_k)$ is surjective when $k \leq 9$. Moreover, its image is generated by permutations of the E_i and the Cremona transformations $R_{ij\ell}$.[2]

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[2] When $k \leq 8$ this is easy to verify directly since $\mathcal{E}(X_k)$ is finite with elements as listed in (2.1). For example, the following composite takes E_1 to $\widehat{E}_1 := 3L - 2E_1 - E_{234567}$:

$$E_1 \ \xrightarrow{R_{123}} \ L - E_{23} \ \xrightarrow{R_{145}} \ 2L - E_{12345} \ \xrightarrow{R_{167}} \ 3L - 2E_1 - E_{234567}.$$

Consider the following elements of $H_2(X_7; \mathbb{Z})$:

$$(2.2) \qquad \begin{aligned} \varepsilon_7 : \ &= \ \tfrac{1}{12}\big(6L - 2E_{123} - 3E_{4567}\big), \\ \widehat{L} : \ &= \ 8L - 3E_{1\dots7}, \\ \widehat{E}_i : \ &= \ 3L - 2E_i - \sum_{j \neq i} E_j, \quad i = 1, \dots, 7; \end{aligned}$$

and of $H_2(X_8; \mathbb{Z})$:

$$(2.3) \qquad \begin{aligned} \varepsilon_8 : \ &= \ \tfrac{1}{30}\big(15L - 5E_{123} - 6E_{45678}\big), \\ \widetilde{L} : \ &= \ 17L - 6E_{1\dots8} \\ \widetilde{E}_i : \ &= \ 6L - 3E_i - 2\sum_{j \neq i} E_j \quad i = 1, \dots, 8. \end{aligned}$$

Proposition 2.1. *For $k = 7, 8$, there is a diffeomorphism $\psi : X_k \to X_k$ in $\mathrm{Diff}_K(X_k)$ that takes the classes L, E_i to $\widehat{L}, \widehat{E}_i$ when $k = 7$ and to $\widetilde{L}, \widetilde{E}_i$ when $k = 8$. Moreover $\psi_*(\varepsilon_k) = -\varepsilon_k$.*

Proof. By the results of Wall mentioned above, it suffices to prove that there is an element of $\mathrm{Aut}_K(X_k)$ with this action. But $H_2(X_k)$ is generated by the classes L, E_i with relations

$$L^2 = 1 = -E_i^2, \quad L \cdot E_i = E_i \cdot E_j = 0 \quad \text{if } i \neq j.$$

Further K is determined by the identities $K \cdot L = -3, K \cdot E_i = -1$. Therefore to prove the first statement in the case $k = 7$, one simply needs to check that the following identities hold for all $1 \leq i, j \leq 7$:

$$\widehat{L}^2 = 1, \ \ \widehat{E}_i \cdot \widehat{E}_j = -\delta_{ij}, \ \ \widehat{L} \cdot \widehat{E}_i = 0, \ \ K \cdot \widehat{E}_i = -1, \ \ K \cdot \widehat{L} = -3.$$

A similar argument works when $k = 8$.

The last statement holds because

$$(2.4) \qquad -\varepsilon_7 = \tfrac{1}{12}\big(6\widehat{L} - 2\widehat{E}_{123} - 3\widehat{E}_{4567}\big), \quad -\varepsilon_8 = \tfrac{1}{30}\big(15\widetilde{L} - 5\widetilde{E}_{123} - 6\widetilde{E}_{45678}\big).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \square$

Remark 2.2. (i) As we shall see in §4, there are other possibilities for ψ. However, they all involve classes of the type $\widehat{E}_i$ and $\widetilde{E}_j$. As is shown in the proof of Proposition 1.5 in [17], these are precisely the classes that give the obstructions to embedding $\lambda E(1, \ell)$ into $E(2, 3)$ for large λ. Hence their size must decrease to 0 as one approaches the critical value $\kappa = \ell$ from below, so that they are natural candidates for the classes of the exceptional divisors created as κ decreases through ℓ.

(ii) For sufficiently generic complex structures on X_k one can choose the map ψ to be a biholomorphic involution. When $k = 7$ one gets the family of Geiser involutions, while when $k = 8$ one gets the Bertini involutions. They may be recognized by the fact that in each case the sum $A + \psi_*(A)$ for $A \in H_2(X_k)$ is always a multiple of the canonical class $K = -3L + \sum E_i$; cf. Dolgachev–Iskovskikh [3]. No doubt one could

use this fact to construct complex structures on M_ℓ. But because we are interested in the singular complex structures on X_k that are pulled back from Z, one would need to look at the moduli spaces of these involutions quite carefully. In §4 we shall take a somewhat different approach.

2.2. **Resolving weighted projective spaces.** We first describe the reduced manifolds (Z, ω). Since these are weighted blow ups of weighted projective spaces, we shall begin with some background information on these spaces. For further details, see Godinho [6].

Let $\underline{m} := (m_1, \ldots, m_N)$ where the m_i are positive integers. Denote $a_i := \prod_{j \neq i} m_j$ and $A := \prod m_i$, so that $a_i m_i = A$ for all i. By definition, the weighted projective space $W := \mathbb{C}P_{\underline{m}}^{N-1}$ is the complex orbifold obtained by quotienting $\mathbb{C}^N \smallsetminus \{0\}$ by the group $\mathbb{C}^*$ acting via

$$\lambda \cdot (z_1, \ldots, z_N) = (\lambda^{m_1} z_1, \ldots, \lambda^{m_N} z_N).$$

We shall normalize the symplectic form ω_0 on $\mathbb{C}^N$ so that the Hamiltonian function for the induced Hamiltonian action of $S^1 \subset \mathbb{C}^*$ on $(\mathbb{C}^N, \omega_0)$ is

$$H_{\underline{m}} := \sum m_i |z_i|^2 = A\left(\sum \frac{|z_i|^2}{a_i}\right).$$

Then $\mathbb{C}P_{\underline{m}}^{N-1}$ may also be considered as one of the reduced spaces of this action and given the corresponding symplectic form $\tau_{\underline{m}}$. To keep our coefficients integral, we shall identify it with the reduced space at level A. Thus

$$(\mathbb{C}P_{\underline{m}}^{N-1}, \tau_{\underline{m}}) = H^{-1}(A)/S^1$$

is the quotient of the boundary of the ellipsoid

$$E(\underline{a}) := \left\{ z \mid \sum \frac{|z_i|^2}{a_i} \leq 1 \right\} \subset \mathbb{C}^N$$

by the characteristic flow. Note that, for any $c > 0$, the rescaled space $(\mathbb{C}P_{\underline{m}}^{N-1}, c\tau_{\underline{m}})$ is the similar quotient of the boundary $H^{-1}(cA)$ of

$$(2.5) \qquad\qquad c\, E(\underline{a}) := \left\{ z \mid \sum \frac{|z_i|^2}{a_i} \leq c \right\}.$$

By construction, the weighted projective space W is a toric manifold whose moment polytope Δ_W can be identified with the intersection of the hyperplane $\sum \frac{x_i}{a_i} = 1$ with the positive quadrant $\{x_i \geq 0\}$ in $\mathbb{R}^N$. If $m_1 = 1$ then the vertex $(1, 0, \ldots, 0)$ of Δ_W is smooth, and there is an integral affine transformation of $\mathbb{R}^N$ that takes this vertex to 0 and takes Δ_W to the polytope

$$\Delta_{a_2, \ldots, a_N} := \{\underline{x} \in \mathbb{R}^{N-1} \mid x_2, \ldots, x_N \geq 0, \ \sum_{i > 1} \frac{x_i}{a_i} \leq 1\}.$$

Therefore, in this case we can think of W as the compactification of the interior of the ellipsoid $E(a_2, \ldots, a_N)$ that is obtained by adding the quotient of the boundary in which each orbit of the characteristic flow is collapsed to a point.

Example 2.3. Let us specialize to the case $N = 3$. If $\underline{m} = (1, p, q)$, then $a_1 = A = pq, a_2 = q$ and $a_3 = p$. Therefore the moment polytope Δ_W of $W := \left(\mathbb{C}P^2_{1,p,q}, \tau_{1,p,q}\right)$ is the triangle $T_{q,p}$ in $\mathbb{R}^2$ with vertices $(0,0), (q,0)$ and $(0,p)$; see Figure 3.3 and [17].[3] (The fact that the weights q, p of the ellipsoid coincide modulo order with the initial weights $m_i, i > 1$, is an accident that happens in this dimension only.)

As always, this moment polytope determines the symplectic form $\tau_{p,q}$: indeed, for every edge ϵ of the moment polytope Δ_W, the integral of $\tau_{p,q}$ over ϵ equals the *affine length* of ϵ. This can be measured as follows. Take any affine transformation A of $\mathbb{R}^2$ that preserves the integer lattice and is such that $A(\epsilon)$ lies along the x-axis, and then measure the Euclidean length of $A(\epsilon)$. Thus if ϵ has rational slope and endpoints on the integer lattice, $\alpha(\epsilon) = k + 1$ where k is the number of points of the integer lattice in the interior of ϵ. In particular, if p, q are mutually prime,

$$(2.6) \qquad \int_{\mathbb{C}P^1_{p,q}} \tau_{p,q} = 1.$$

The following lemma is due to Tolman [29]. We explain its proof for the convenience of the reader. Note that she uses the form $\omega_{1,m,n} := \frac{1}{mn}\tau_{1,m,n}$ on $\mathbb{C}P^2_{1,m,n}$.

Lemma 2.4 (Tolman). *Suppose that the manifold M_ℓ exists for some integer $\ell \in [2, 5]$. Then the reduced space (Z, ω_κ) at level $\kappa \in (-\ell, \ell)$ is diffeomorphic to the connected sum $\mathbb{C}P^2_{1,2,3}\#\overline{\mathbb{C}P}^2_{1,1,\ell}$ of the weighted projective space $\mathbb{C}P^2_{1,2,3}$ with a conjugate $\overline{\mathbb{C}P}^2_{1,1,\ell}$. Moreover the symplectic form ω_κ lies in the unique class $[\omega_\kappa]$ such that*

$$(2.7) \qquad [\omega_\kappa]|_{\mathbb{C}P^1_{2,3}} = \tfrac{6+\kappa}{6}\,\tau_{2,3}, \quad [\omega_\kappa]|_{\mathbb{C}P^1_{1,\ell}} = \tfrac{\ell+\kappa}{\ell}\,\tau_{1,\ell}.$$

Proof. It follows from equation (2.5) that the reduced space for the Hamiltonian $H := \sum_{i=1}^3 m_i|z_i|^2$ at level $\varepsilon > 0$ is $\left(\mathbb{C}P^2_{\underline{m}}, \frac{\varepsilon}{A}\,\tau_{\underline{m}}\right)$. Thus, if $\underline{m} = (1, 2, 3)$ the reduced space is

$$\left(\mathbb{C}P^2_{1,2,3}, \tfrac{\varepsilon}{6}\,\tau_{1,2,3}\right).$$

Since the minimal critical level is at $\kappa = -6$ rather than 0, the coefficient of $\tau_{2,3}$ in equation (2.7) is therefore $\frac{6+\kappa}{6}$.

To understand the diffeomorphism type of the reduced space at level $\kappa \in (-\ell, \ell)$, first recall from Example 2.3 that when $\underline{m} = (1, m_2, m_3) =: (1, \underline{m}')$, one can also obtain $(\mathbb{C}P^2_{\underline{m}}, \tau_{\underline{m}})$ from the ellipsoid $E := E(\underline{m}') \subset \mathbb{C}^2$ by collapsing its boundary ∂E to $\mathbb{C}P^1_{\underline{m}'}$ as above. It follows that the connected sum $X\#\overline{\mathbb{C}P}^2_{1,\underline{m}'}$ can be considered as a orbifold blow up, in which one cuts out an embedded ellipsoid $\varepsilon E(\underline{m}') \subset X$ for some small $\varepsilon > 0$ and then collapses the boundary along the characteristic flow. This is called the (symplectic) orbifold blow up with weights $\underline{m}'$. Using toric models one can show that as one passes a critical point with isotropy weights $(-1, m_2, m_3)$ (where $m_i > 0$) the critical level undergoes an orbifold blow up with weights $\underline{m}'$. This is illustrated in Figure 3.1 below, and a detailed proof is given by Godinho [6].

---

[3] For a general treatment of toric symplectic orbifolds see Lerman and Tolman [13].

For example, the reduced space at level $\varepsilon > 0$ of the function $H = -|z_1|^2 + m_2|z_2|^2 + m_3|z_3|^2$ has as exceptional divisor the quotient of the level set

$$H(z_2, z_3) = m_2 m_3 \left(\frac{|z_2|^2}{m_3} + \frac{|z_3|^2}{m_2} \right) = \varepsilon,$$

which is $\left(\mathbb{C}P^1_{m_3, m_2}, \frac{\varepsilon}{m_2 m_3} \tau_{m_3, m_2} \right)$. In particular, when $(m_2, m_3) = (1, \ell)$ and the critical point occurs at level $-\ell$, one obtains the coefficient $(\ell + \varepsilon)/\ell$ of (2.7). $\qquad\square$

Remark 2.5. When $\kappa + \ell > 0$ is sufficiently small the weighted blow up can be done equivariantly so that (Z, ω_κ) has a global toric structure as in Figure 2.2. We shall denote by J_T the corresponding complex structure on Z.

Observe that Z has three singular points $p_m, m = 2, 3, \ell$, each with a neighborhood $\mathcal{N}_m$ of the form $\widetilde{\mathcal{N}}_m / \mathbb{Z}_m$, where $\widetilde{\mathcal{N}}_m := B \subset \mathbb{C}^2$ is a (closed) ball with suitable small radius and the generator of $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ acts via $(z_1, z_2) \mapsto (e^{2\pi i/m} z_1, e^{-2\pi i/m} z_2)$.[4] For example, $\mathcal{N}_3$ is a neighborhood of $[0 : 0 : 1]$ in $\mathbb{C}P^2_{1,2,3}$ and $\lambda \in \mathbb{Z}_3$ acts by

$$[z_0 : z_1 : 1] \quad \mapsto \quad [\lambda z_0 : \lambda^2 z_1 : \lambda^3] = [\lambda z_0 : \lambda^{-1} z_1 : 1].$$

We shall denote $\mathbf{p} := \{p_2, p_3, p_\ell\}$ and $\mathcal{N} := \cup_m \mathcal{N}_m$. By the equivariant Darboux theorem we may (and will) suppose that any symplectic form ω on Z lifts to the standard form $\widetilde{\omega}_0 := \sum_j dx_j \wedge dy_j$ on the local uniformizers $\widetilde{\mathcal{N}}_m$, where $z_j := x_j + iy_j$.

Although Z can be given an orbifold structure, it is better to think of it as a manifold with singular points. Since the order of these singularities are different, any diffeomorphism of Z must fix each p_m. Then the condition for $\phi : Z \to Z$ to be a diffeomorphism is that its restriction to the manifold $Z \setminus \mathbf{p}$ is smooth and that for each m there is an open, $\mathbb{Z}_m$-invariant neighborhood $\widetilde{U}_m$ of 0 in $\widetilde{\mathcal{N}}_m$ and a diffeomorphism $\widetilde{\phi}_m : (\widetilde{U}_m, \widetilde{\omega}_0) \to (\widetilde{\mathcal{N}}_m, \widetilde{\omega}_0)$ that takes $\mathbb{Z}_m$-orbits to $\mathbb{Z}_m$-orbits; i.e. the following diagram commutes

$$
(2.8) \qquad
\begin{array}{ccc}
\widetilde{U}_m & \overset{\widetilde{\phi}_m}{\to} & \widetilde{\mathcal{N}}_m \\
\downarrow & & \downarrow \\
U_m & \overset{\phi}{\to} & \mathcal{N}_m.
\end{array}
$$

Standard arguments show that any diffeomorphism can be isotoped to one that is linear with respect to these local coordinates near $\mathbf{p}$. Hence we shall assume that the $\widetilde{\phi}_m$ are linear. It is then clear that for each m there is an automorphism $\alpha_m : \mathbb{Z}_m \to \mathbb{Z}_m$ such that

$$(2.9) \qquad \widetilde{\phi}_m \circ \gamma = \alpha_m(\gamma) \circ \widetilde{\phi}_m, \quad \gamma \in \mathbb{Z}_m.$$

Similarly, a diffeomorphism $\phi : (Z, \omega) \to (Z, \omega')$ is called a *symplectomorphism* if its restriction to the manifold $(Z \setminus \mathbf{p}, \omega)$ is a symplectomorphism, and if the local lifts $\widetilde{\phi}_m$ preserve ω_0.

---

[4] These are known in the literature as simple singularities of type A_{m-1}: see for example Ohta–Ono [23]. They may be resolved by chains of -2spheres of length $m - 1$; cf. Lemma 2.7.

Note finally that because we are thinking of Z as a singular space, rather than as an orbifold, we define its homology and cohomology groups to be those of the underlying topological space.

Lemma 2.6. (i) *Every diffeomorphism* $(Z, \omega) \to (Z, \omega')$ *is isotopic to a diffeomorphism* ϕ *such that each local linear model* $\widetilde{\phi}_m$ *is either the identity map or, when* $m = 3, \ell$, *has the form* $(z_1, z_2) \mapsto (z_2, z_1)$.

(ii) *Denote by* $e(Y) \in H^2(Z \setminus \mathbf{p}; \mathbb{Z})$ *the Euler class of an* S^1*-bundle* $Y|_{Z \setminus \mathbf{p}} \to Z$. *If* $\psi^*(e(Y)) = -e(Y)$, *then* ϕ *has the local model* $(z_1, z_2) \mapsto (z_2, z_1)$ *for* $m = 3, \ell$, *but if* $\phi^*(e(Y)) = e(Y)$ *then* ϕ *is locally modelled by the identity map.*

(iii) *The above statements hold also for symplectomorphisms.*

Proof. (i) follows from the above discussion because there is only one nontrivial equivariant automorphism α_m of $\mathbb{Z}_m$ when $m = 3, \ell$, namely $\gamma \mapsto \gamma^{-1}$, while there are none for $m = 2$. (ii) holds because $\phi^*(e(Y)) = -e(Y)$ only if ϕ induces the nontrivial automorphism on $\mathbb{Z}_m$ for $m = 3, \ell$, while $\phi^*(e(Y)) = e(Y)$ only if the induced automorphisms on $\mathbb{Z}_m$ are trivial. The proof of (iii) is similar. $\square$

To go further, we need to consider the relation between Z and its resolution X_k, where $k = \ell + 3$. We construct the complex manifold (X_k, J) from $\mathbb{C}P^2$ by blowing up k times (in the complex category)[5] as follows.

Roughly speaking X_k is obtained by blowing $\mathbb{C}P^2$ up three times at one point p and ℓ times at another point q. However there are several inequivalent ways of doing this. By blowing up repeatedly at some point p we mean the following: blow up at $p =: p_1$ creating an exceptional divisor C_{E_1} in class E_1, then blow up at some point $p_2 \in C_{E_1}$ obtaining a new exceptional divisor C_{E_2} in class E_2 and the proper transform $C_{E_1 - E_2}$ of C_{E_1}, and continue, at the ith stage blowing up at some point p_i on the exceptional divisor $C_{E_{i-1}}$ to obtain C_{E_i} and $C_{E_{i-1} - E_i}$. We shall only consider the case when $p_{i+1} \notin C_{E_{i-1} - E_i}$ so that the blowing up process results in a chain of intersecting -2 curves in the classes $E_1 - E_2, E_2 - E_3, \ldots$. Even so, this process is not unique: although there is only one way of doing this twice, there is a choice at the third blow up. To see this, suppose that C_L is the unique line in $\mathbb{C}P^2$ through p_1 and with proper transform $C_{L - E_1}$ through p_2. Then its proper transform after the second blow up is $C_{L - E_1 - E_2}$, which intersects C_{E_2} at one point. If we choose p_3 to be this point of intersection, the third blow up contains curves C_1, C_2, C_3, C_0 in classes $E_1 - E_2, E_2 - E_3$, E_3 and $L - E_{123}$, respectively. In this case we shall say that the blow up at p is directed by C_L: all such blow ups are locally biholomorphic since they depend only on p and C_L. More generally, if Q is an embedded (perhaps noncompact) holomorphic curve through p, we say that repeated blow ups at p are *directed by* Q if we always choose the blow up point $p_i \in C_{E_{i-1}}$ to lie on the proper transform of Q.

---

[5] In this paper, there is constant interplay between complex and symplectic blowing up; the former procedure replaces a point by the family of complex lines through that point, while in the latter replaces a ball or ellipsoid by the curve obtained by collapsing its boundary.

When constructing the resolution (X_k, J) as a blow up, we always assume that the 3-fold blow up at p is directed by a line C_L, and that the ℓ-fold blow up at q is generic with respect to p, C_L. In other words, we assume $q \notin C_L$, and also choose the center q_2 of the second blow up not on the proper transform C' of the line through p, q so that C' (which lies in class $L - E_{14}$) lifts to (X_k, J). For the moment we make no further restrictions on the blow up at q (though we will do this in §4). Therefore, besides the curves $C_0, \dots, C_3$ mentioned above, (X_k, J) contains holomorphic curves $C_4, \dots, C_{k-1}, C_k$ in classes $E_4 - E_5, \dots, E_{k-1} - E_k, E_k$ respectively. We denote by $\mathcal{C}$ the set of curves $C_i, 0 \le i \le k-1, i \ne 3$: cf. Fig 2.1. (The curve C' is irrelevant for now, but appears in the proof of Lemma 2.22.)
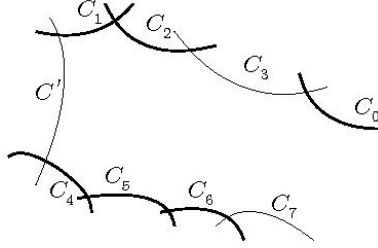
FIGURE 2.1. The curves in $\mathcal{C}$ for the case $k = 7$, together with C_3, C_7 and the curve C' in class $L - E_{14}$.

Note that all the curves in $\mathcal{C}$ have self-intersection -2, and belong to one of three connected components, $C_0, C_1 \cup C_2$, and $C_4 \cup \cdots \cup C_{k-1}$. It is well known that a string of -2 curves of length s blows down to a simple singularity of order $s + 1$ and type A_s. Thus C_0 gives a point of order 2, $C_1 \cup C_2$ a point of order 3 and $C_4 \cup \cdots \cup C_{k-1}$ a point of order ℓ. Hence the blow down of (X_k, J) that contracts these curves gives an orbifold with the same singularities as Z.
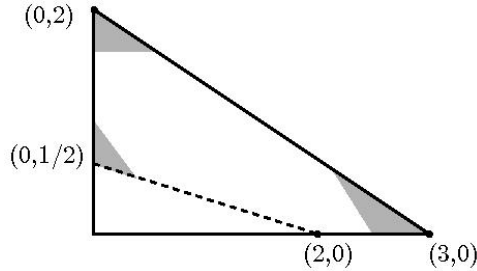
FIGURE 2.2. $\frac{1}{2}T_{1,4}$ embedded in $T_{2,3}$ (where $\lambda T_{a,b}$ denotes the triangle with vertices $(0,0), (0,\lambda a), (\lambda b, 0)$.) $\Delta := \Delta(\frac{1}{2})$ is defined to be $T_{2,3} \setminus \mathrm{int}\left(\frac{1}{2}T_{1,4}\right)$. The shaded regions in Δ form the image of the neighborhood $\mathcal{N}$ of the singular points of Z.

Lemma 2.7. *The complex orbifold obtained from (X_k, J) by contracting the three components of $\mathcal{C}$ is $Z := \mathbb{C}P^2_{1,2,3} \# \overline{\mathbb{C}}P^2_{1,1,\ell}$.*

Proof. We shall assume that $\ell = 4$ for simplicity. The case $\ell = 5$ is similar. Denote by $\lambda T_{a,b}$, where $a < b$, the triangle in $\mathbb{R}^2$ with vertices $(0,0), (0, \lambda a), (\lambda b, 0)$; see Figure 2.2. As explained in Example 2.3 the triangle $T_{2,3}$ is the the moment polytope for the weighted projective space $(\mathbb{C}P^2_{1,2,3}, \tau_{1,2,3})$. Similarly, the complement $\Delta := \Delta(\lambda)$ of int $(\lambda T_{1,4})$ in $T_{2,3}$ is the moment polytope of its blow up $(Z, c\omega_\kappa)$ where $\lambda := \frac{3(\kappa+4)}{2(\kappa+6)}$ and $c := \frac{6}{\kappa+6}$. (These constants can be worked out from equations 2.6 and 2.7.)

Fulton explains in [5] how to resolve the singularities of a toric orbifold by blowing up. Because he is working in the complex rather than symplectic category, he describes the toric variety by its fan (the set of conormals to the facets); the process of blowing up adds extra elements to the fan. One can check that the resulting fan is precisely that of the "approximation" Δ_ε to Δ that is illustrated in Figure 2.3 below. Here the edges going clockwise from C_1^ε have outward conormals:

$$(0,1), \ (1,2), \ (2,3)^*, \ (1,1), \ (0,-1)^*,$$
$$(-1,-4)^*, \ (-1,-3), \ (-1,-2), \ (-1,-1), \ (-1,0)^*,$$

where the starred vectors are also conormals of Δ. Therefore Δ_ε is smooth (i.e. the determinant of any successive pair of edges has absolute value 1), and it is easy to check that each of its short edges C_i^ε represents a sphere with self intersection -2 (where $i = 0, 1, 2, 4 \ldots, 7$). The corresponding symplectic toric manifold is a symplectic version of (X_7, J), with the curve C_i in $\mathcal{C}$ identified to the short edge C_i^ε. Thus the resolution described by Fulton is precisely (X_k, J). $\square$

Definition 2.8. *For any complex structure J on X_k constructed as above, we shall call the holomorphic blow down map $\Phi_J : (X_k, J) \to Z$ the **resolution** of Z. Further, we denote by $\mathcal{C}$ the collection of curves $C_i, 0 \leq i < k, i \neq 3$, in X_k, and by*

$$D_1 := \Phi_J(C_3) \cong \mathbb{C}P^1_{2,3}, \quad D_2 := \Phi_J(C_k) \cong \mathbb{C}P^1_{1,\ell}$$

the two divisors in Z.

Note that Φ_J is bijective outside the singular points and contracts each connected component of $\mathcal{C}$ to one of the singular points of Z. We shall say more about the resolution as a symplectic manifold later. For now we shall use it to understand the (co)homology of Z.

Lemma 2.9. *For $\ell = 4, 5$, $H^1(Z \setminus \mathbf{p})$ is a free abelian group. Moreover, there is a commutative diagram with exact rows*

$$
\begin{array}{ccccccccc}
0 & \to & H^2(Z; \mathbb{Z}) & \to & H^2(Z \setminus \mathbf{p}; \mathbb{Z}) & \to & H^2(\mathcal{N} \setminus \mathbf{p}; \mathbb{Z}) & \to & 0 \\
& & \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \\
0 & \to & \mathbb{Z}^2 & \xrightarrow{\alpha} & \mathbb{Z}^2 & \to & \mathbb{Z}_6 \oplus \mathbb{Z}_\ell & \to & 0,
\end{array}
$$

where the maps in the top row are induced by restriction and where $\alpha(m,n) = (6m, \ell n)$.

Proof. We shall prove this for the case $\ell = 4$ and then indicate the few changes that need to be made when $\ell = 5$. We shall calculate $H^*(Z) := H^*(Z; \mathbb{Z})$ by comparing Z with its resolution X_7.

Denote by $V \subset X := X_7$ the inverse image $\Phi_J^{-1}(\mathcal{N})$ where Φ_J is as in Definition 2.8. Then $\partial V \cong \partial \mathcal{N}$ is a disjoint union of three lens spaces and hence has $H_1(\partial V) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$, while $H_2(\partial V) = 0$. Thus from the Mayer-Vietoris sequence of the decomposition $X = (X \smallsetminus V) \cup V'$ (where $V' \supset V$ is a slight enlargement of V) we obtain the exact sequence

$$0 \to H_2(X \smallsetminus V) \oplus H_2(V) \to H_2(X) \to H_1(\partial V) \to H_1(X \smallsetminus V) \to 0,$$

where we use integral coefficients. Now $H_2(V)$ is generated by $E_1 - E_2, E_2 - E_3, L - E_{123}, E_4 - E_5, E_5 - E_6, E_6 - E_7$, while $H_2(X \smallsetminus V)$ is generated by those elements of $H_2(X)$ that are orthogonal to $H_2(V)$ with respect to the intersection pairing. Thus $3L - E_{123}$ and E_{4567} form a generating set for $H_2(X \smallsetminus V)$. Hence $L, L - E_3, E_7 \in H_2(X)$ project to elements in the quotient $H_1(\partial V)$ of orders $2, 3, 4$ respectively. Thus the map $H_2(X) \to H_1(\partial V)$ is surjective, so that $H_1(X \smallsetminus V) = 0$.

Now consider the commutative diagram induced by Φ_J:

$$
\begin{array}{ccccccccc}
0 & \to & H_2(X \smallsetminus V) \oplus H_2(V) & \to & H_2(X) & \to & H_1(\partial V) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_2(Z \smallsetminus \mathcal{N}) \oplus H_2(\mathcal{N}) & \to & H_2(Z) & \to & H_1(\partial \mathcal{N}) & \to & 0
\end{array}
$$

Since Φ_J is a homeomorphism $X \smallsetminus V \to Z \smallsetminus \mathcal{N}$ and $H_i(\mathcal{N}) = 0$ for $i > 0$, we see that $H_2(Z)$ is generated by elements $(\Phi_J)_* E_3, (\Phi_J)_* E_7$, while the image of $H_2(Z \smallsetminus \mathcal{N})$ is generated by $6(\Phi_J)_* E_3$ (from $3L - E_{123}$) and $4(\Phi_J)_* E_7$ (from E_{4567}).

Since $X \smallsetminus V \cong Z \smallsetminus \mathcal{N}$, we know from above that $H_1(Z \smallsetminus \mathcal{N}) = 0$. A similar Mayer-Vietoris sequence argument shows that $H_1(Z) = 0$. Hence $H^2(Z \smallsetminus \mathcal{N})$ and $H^2(Z)$ are both free abelian groups and the map between them is dual to the inclusion $H_2(Z \smallsetminus \mathcal{N}) \to H_2(Z)$. This completes the proof when $\ell = 4$.

When $\ell = 5$ one just needs to add a further blow up to the chain $E_4, \ldots, E_7$. Thus the generators of $H_2(X \smallsetminus V)$ are $3L - E_{123}$ and $E_{4\ldots 8}$. The rest of the argument is essentially the same. $\qquad\square$

Corollary 2.10. *Suppose that M_ℓ exists and denote $Y := H^{-1}(0)$ considered as the boundary of $H^{-1}([-6, 0])$. Then there is a unique class $e_Z(Y) \in H^2(Z; \mathbb{Q})$ that restricts to the Euler class $e(Y)$ of the locally trivial S^1-bundle $Y|_{Z \smallsetminus \mathbf{p}} \to Z \smallsetminus \mathbf{p}$. Moreover if we identify Z with $\mathbb{C}P^2_{1,2,3} \# \overline{\mathbb{C}P}^2_{1,1,\ell}$ as in Lemma 2.4 then*

$$(2.10) \qquad [e_Z(Y)]|_{\mathbb{C}P^1_{2,3}} = -\tfrac{1}{6}\,\tau_{2,3}, \quad [e_Z(Y)]|_{\mathbb{C}P^1_{1,\ell}} = -\tfrac{1}{\ell}\,\tau_{1,\ell}.$$

Proof. The first statement is an immediate consequence of Lemmas 2.4 and 2.9. The second also uses the fact that $e_Z(Y) = -\frac{d}{d\kappa}[\tau_\kappa]$ by Godinho's generalization of the Duistermaat–Heckmann formula. Note that $e_Z(Y)$ does restrict to an integral class on $Z \smallsetminus \mathbf{p}$ because the image of $H_2(Z \smallsetminus \mathbf{p})$ in $H_2(Z)$ is generated by $6E_3, 4E_7$. $\qquad\square$

2.3. **The symplectic topology of the reduced spaces: preliminaries.** By Corollary 2.10, the reduced space (Z, ω_κ) for $\kappa \in (-\ell, \ell)$ is an orbifold blow up. It can be constructed as a toric manifold whenever

$$\ell + \kappa < 3 + \kappa/2, \quad \text{or equivalently } -\ell < \kappa < 2(3 - \ell)$$

since then the triangle $(1 + \kappa/\ell)T_{1,\ell}$ is a subset of $(1 + \kappa/6)T_{2,3}$.

The following lemma is proved in [17, Prop 1.6]. (The argument is explained below.)

Lemma 2.11. *For all integers $\ell \in [2, 6]$ and all $\kappa \in (-\ell, \ell)$ there are symplectic orbifolds (Z, ω_κ) satisfying the conditions in Lemma 2.4.*

Remark 2.12. In fact, if all we are interested in is existence then we do not need this result from [17] because $[\omega_0]$ is the anticanonical class $-K$ on Z. Hence, provided that we give Z a sufficiently generic complex structure J_Z, we can take ω_0 to be the Kähler form induced from projective space by the embedding given by sections of a suitable multiple of the anticanonical class; and then define ω_κ for $-\ell < \kappa < 0$ by decreasing the size of the exceptional divisor $\mathbb{C}P^1_{1,4}$, or, equivalently, by decreasing the size of the ellipsoid $\lambda E(1, 4)$ that is embedded in $E(2, 3)$. This constructs (Z, ω_κ) for $-\ell < \kappa \leq 0$. The result for $\kappa > 0$ follows by symmetry. More precisely, we will see in the proof of Proposition 1.3 given in §3 below that the diffeomorphism ψ of Proposition 2.1 covers a diffeomorphism ψ_Z of Z such that $\psi_Z^*([\omega_\kappa]) = -[\omega_{-\kappa}]$.

But notice that we do need J_Z to be "generic". In particular we cannot use the toric structure J_T because this is not NEF; for example when $\ell = 4$ the edge in Figure 2.3 with conormal $(0, -1)$ pulls back to a line in X_7 in class $L - E_{4567}$ and $K \cdot (L - E_{4567}) = 1$. Suitable complex structures are constructed in Lemmas 4.2 and 4.3.

The above remarks, together with Proposition 2.20 below, are all that is needed to construct M_ℓ as a symplectic, or indeed as a complex, manifold. However, to establish the uniqueness results, we need to know much more about the reduced spaces Z than simply the existence of suitable symplectic forms. We now adapt a definition from Gonzalez [7]. The word "rigid" is used here by analogy with the complex case. Symplectic forms can always be deformed, but in the rigid case these deformations have very little consequence, and the symplectic structure is essentially unique.

Definition 2.13. *A symplectic orbifold (Z, ω) is called **rigid** if the following conditions hold:*

(a) *(**Uniqueness.**) Any two cohomologous symplectic forms on Z are diffeomorphic;*

(b) *(**Deformation implies isotopy.**) Every path $\omega_t, t \in [0, 1]$, of symplectic forms on Z with $[\omega_1] = [\omega_0]$ can be homotoped through families of symplectic forms with the fixed endpoints ω_0 and ω_1 to an isotopy, i.e. a path ω_t' such that $[\omega_t']$ is constant;*

(c) *(**Connectness.**) For all symplectic forms ω' on Z the group $\mathrm{Symp}_H(Z, \omega')$ of symplectomorphisms that act trivially on integral homology is connected.*

Note that condition (c) implies that the diffeomorphism in (a) is determined uniquely up to symplectic isotopy by its action on $H_*(Z; \mathbb{Z})$.

Remark 2.14. To put our results on Z in perspective, we observe that the papers [10, 15] show that (X_k, ω) satisfies the first two of these conditions; it satisfies (c) when $k \leq 3$ (cf. Lalonde–Pinsonnault [12] for the case $k \leq 2$ and Pinsonnault [25] for $k = 3$), the case $k = 4$ is open, but when $k \geq 5$ Seidel showed in [27] that there are ω' on X_k for which (c) does not hold, by constructing symplectomorphisms that twist one Lagrangian sphere around another. But none of these Dehn twists can be constructed so as to descend to Z. Hence the nonrigidity of these X_k does not contradict the rigidity of Z.

Lemma 2.15. *The weighted projective space $\mathbb{C}P^2_{1,2,3}$ is symplectically rigid.*

Proof. Condition (b) is obviously satisfied since we can make arbitrary changes in the cohomology class by rescaling. The other conditions can be proved by adapting the arguments given below for Z. Further details are left to the reader. $\qquad\square$

The proof of the following theorem takes up the rest of this section.

Theorem 2.16. *The orbifold $Z = \mathbb{C}P^2_{1,2,3} \# \overline{\mathbb{C}P}^2_{1,1,\ell}$ is symplectically rigid.*

We prepare for the proof by collecting together some useful technical results.

FIGURE 2.3. The toric model for the curves C_i, where $0 \leq i < 7, i \neq 3$, in $\mathcal{C}$ when $\ell = 4$, where we denote the edge representing (C_i, τ_ε) by C_i^ε. The moment image in Δ of the neighborhood $\mathcal{N}$ is shaded as in Figure 2.2; τ_ε is the corresponding form on the small neighborhood $\mathcal{V}$ of $\mathcal{C}$.

The problem with the resolution $\Phi_J : (X_k, J) \to Z$ from the symplectic point of view is that Φ_J is not a symplectomorphism; in particular, the pull back $\Phi_J^*(\omega)$ of any symplectic form on Z is degenerate along $\mathcal{C}$. However, we can deal with this as in [17], replacing $\Phi_J^*(\omega)$ by a symplectic approximation as follows.

By the local Darboux theorem explained before Lemma 2.6 every symplectic form on Z can be isotoped to be standard in the neighborhoods $\mathcal{N}_m$ of the singular points. Therefore, we shall only consider symplectic forms on Z that are standard in $\mathcal{N} := \cup \mathcal{N}_m$. Since the standard form is toric, we may identify the neighborhood of each singular point with a neighborhood of the appropriate vertex in the toric model Δ

illustrated in Figure 2.2 for the case $\ell = 4$. The pullback of this standard form by Φ_J is degenerate along $\mathcal{C}$ but is toric elsewhere in $\mathcal{V}$, and clearly may be modified inside $\mathcal{V}$ to a form

$$\tau_\varepsilon$$

that is toric and nondegenerate, and so that its (local) moment polytope is a neighborhood of the short edges C_i^ε in the approximation Δ_ε. Figure 2.3 illustrates the case $\ell = 4$.[6] There is an analogous picture for $\ell = 5$ with one extra short edge with conormal $(-1, -5)$. The paper [17] describes to how to construct such a symplectic approximation for any orbifold blow up of $\mathbb{C}P^2_{1,a,b}$. In the language of that paper we are replacing the curves in $\mathcal{C}$ by the relevant parts of the inner and outer approximations to the ellipsoids $\lambda_1 E(2,3)$ and $\lambda_2 E(1,\ell)$.

Here are some useful properties of τ_ε:

- As $\varepsilon \to 0$, τ_ε converges to $\Phi_J^*(\omega)$ in $\mathcal{V}$.
- The curves $C_i^\varepsilon, C_{i+1}^\varepsilon$ intersect τ_ε-orthogonally. (This is true for the spheres corresponding to any pair of intersecting edges of the moment polytope. For this is obviously true if the two edges lie along the coordinate axes through the origin. But, by the Delzant (smoothness) condition, all vertices are affine equivalent to this one, and affine transformations of the moment polytope lift to symplectomorphisms of the toric manifold.)
- We may recover ω near $\mathbf{p}$ from τ_ε (and hence $\Phi_J^*(\omega)$ near $\mathcal{C}$) by the blowing down process described in Symington [28]; see also [17, Lemma 2.3]. This is a generalized symplectic summing process that first removes the curves $\mathcal{C}$ and then adds a suitable standard contractible open set.

In the following discussion we shall allow ourselves to decrease ε and shrink the sets $\mathcal{N}, \mathcal{V}$ as necessary. Note also that *this local toric model may be extended to include the -1 curves C_3 and C_k*. Thus we shall assume that τ_ε is nondegenerate on C_3 and C_k and that these intersect $\mathcal{C}$ orthogonally with respect to the symplectic form. The symplectic neighborhod theorem then implies that

- τ_ε is uniquely determined near $\mathcal{C} \cup (C_3 \cup C_k)$ by its cohomology class.

Given $\varepsilon > 0$ we shall further assume that $\int_{C_3} \tau_\varepsilon = 1 - 3\varepsilon$, so that τ_ε integrates to 1 over $\Phi^{-1}(D_1) = \cup_{i=0}^3 C_i$. However there is a choice for the size of C_k. If it is necessary to emphasize this, we shall denote the form that integrates to $\lambda - (\ell - 1)\varepsilon$ over C_k by $\tau_{\lambda,\varepsilon}$. Then

$$\int_{\Phi^{-1}(D_2)} \tau_{\lambda,\varepsilon} = \sum_{i=4}^k \int_{C_i} \tau_{\lambda,\varepsilon} = \lambda.$$

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[6] In this figure we are illustrating a case in which (Z, ω) has a global toric structure. This does not hold for all $[\omega_\kappa]$; but all that concerns us here is the local toric structure near the singular points, which always exists.

Thus, in the notation of equation (1.1),

$$[\tau_{\lambda,\varepsilon}] = 3a - (e_1 + e_2 + e_3) - \lambda(\sum_{i=4}^{k} e_k) + O(\varepsilon).$$

We shall also suppose that $0 < \varepsilon \ll \lambda \le 1$. More precisely, we choose $\varepsilon > 0$ so small that

(2.11)   the minimum of $\int_E \tau_{\lambda,\varepsilon}$ for $E \in \mathcal{E}(X_k)$ is assumed on the class E_k.

(That this is possible can be directly checked using the description of $\mathcal{E}(X_k)$ given in equation (2.1).)

Definition 2.17. *We denote by $\tau_{\varepsilon,\lambda}$ (simplified to τ_ε) the toric symplectic form on $\mathcal{V} \cup \mathrm{nbhd}\,(C_3 \cup C_k)$ described above. Fix a τ_ε compatible complex structure $J_\mathcal{V}$ on $\mathcal{V}$ for which the curves in $\mathcal{C}$ are holomorphic, and let Ω be any symplectic form on X_k that equals τ_ε in $\mathcal{V}$. Then we define*

- *$\mathcal{J}_\mathcal{V}(\Omega)$ to be the set of Ω-tame almost complex structures that equal $J_\mathcal{V}$ near $\mathcal{C}$.*
- *$\mathcal{J}_{\mathcal{V},reg}(\Omega)$ to be the subset of $J \in \mathcal{J}_\mathcal{V}(\Omega)$ for which every J-holomorphic curve $u : S^2 \to X_k$ whose image intersects $X_k \setminus \mathcal{V}$ has $c_1(u_*[S^2]) > 0$.*

*Further, we say that a class $A \in H_2(X_k)$ is **smoothly J-representable** if it has a J-holomorphic and smoothly embedded representative.*

Here is the key technical lemma.

Lemma 2.18. (i) *The subset $\mathcal{J}_{\mathcal{V},reg}(\Omega)$ has second category in $\mathcal{J}_\mathcal{V}(\Omega)$ and is path connected. For every $J \in \mathcal{J}_{\mathcal{V},reg}(\Omega)$ the class E_3 is smoothly J-representable.*

(ii) [Pinsonnault] *Suppose that $\Omega|_{\mathcal{C} \cup C_3 \cup C_k} = \tau_{\lambda,\varepsilon}$ for some $\lambda \le 1$. Then, for every $J \in \mathcal{J}_\mathcal{V}(\Omega)$ the class E_k is smoothly J-representable.*

Proof. To prove (i), recall that the moduli space $\mathcal{M}(A, J)$ of J-holomorphic maps $u : S^2 \to X_k$ in class A has expected (real) dimension $4 + 2c_1(A)$ and that, if $\mathcal{M}(A, J)$ is nonempty and consists of regular curves, this must be ≥ 6, the dimension of the reparametrization group. But, standard results (cf. [19, Ch 3]) imply that for each A there is a subset of $\mathcal{J}_\mathcal{V}(\Omega)$ of second category consisting of J for which every A-curve that intersects $X \setminus \mathcal{V}$ is regular. Since the set of classes A is countable, such J lie in the set we have called $\mathcal{J}_{\mathcal{V},reg}(\Omega)$. This proves the first statement in (i). To prove that $\mathcal{J}_{\mathcal{V},reg}(\Omega)$ is path connected, recall that any two elements in $\mathcal{J}_{\mathcal{V},reg}(\Omega)$ can be joined by a generic path consisting of elements J_t for which the cokernel of the linearized Cauchy–Riemann operator D_u has dimension at most 1. Therefore, if $\mathcal{M}(A, J_t) \ne \emptyset$, we have $4 + 2c_1(A) \ge 5$, which implies that $c_1(A) > 0$. Thus $J_t \in \mathcal{J}_{\mathcal{V},reg}(\Omega)$ for all t.

Finally, since E_3 has nonzero Gromov–Witten invariant, it is represented by some J-holomorphic stable map for all $J \in \mathcal{J}_\mathcal{V}(\Omega)$. Let $B_1, \ldots, B_m$ be the classes of its components. Since $c_1(E_3) = 1$, if this stable map is not smooth at least one of these components must have $c_1(B_i) \le 0$. But this is impossible when $J \in \mathcal{J}_{\mathcal{V},reg}(\Omega)$. This proves (i).

We prove statement (ii) by using Lemma 1.2 in Pinsonnault [24], which states that a class $E \in \mathcal{E}_k$ whose symplectic area is minimal among all the classes in $\mathcal{E}_k$ has an smooth J-representative for *all* tame J. This applies here since E_k is such a minimal class by construction. $\qquad\square$

2.4. **The rigidity of Z.**

Lemma 2.19. *Z satisfies condition (b) in the definition of rigidity.*

Proof. We shall prove this for the reduced spaces (Z, ω_κ) where $-\ell < \kappa \leq 0$. The case $0 < \kappa < \ell$ follows by symmetry; cf. Remark 2.12.

Suppose we are given two cohomologous symplectic forms ω_0, ω_1 on Z that are connected by a deformation ω_t. First multiply each ω_t by a suitable constant so its integral over the divisor $D_1 = \mathbb{C}P_{2,3}^1$ is constant and equal to 1. Next, use a parametrized version of the local Darboux theorem of §2 to ensure that each of these forms is standard in some neighborhood of the singular set **p**. Then the pullback family $\Phi_J^*(\omega_t)$ on X_k is constant on the neighborhood $\mathcal{V}$. We claim that by a relative version of the "deformation implies isotopy" result from [15] we can homotop the deformation $\Phi_J^*(\omega_t)$ in X_k to an isotopy, keeping the endpoints fixed and also not changing the forms near $\mathcal{C}$. Once this is done, we can push forward the resulting isotopy by Φ_J to an isotopy in Z.

To establish the claim, several remarks are in order.

- Since we are keeping the forms fixed in $\mathcal{V}$ it does not matter that they are not symplectic along $\mathcal{C}$. (Alternatively, we can can change them in $\mathcal{V}$ to equal τ_ε.)

- One changes a deformation to an isotopy by inflating along certain symplectically embedded curves S. If these curves do not intersect $\mathcal{C}$ then this inflation process will not change the forms near $\mathcal{C}$. The general inflation process is described in [15]; some special cases are described in [17].

- We can insure that the curves S do not intersect $\mathcal{C}$ by choosing them to lie in classes in $H_2(X_k \smallsetminus \mathcal{C})$ and also to be J-holomorphic for some J for which the curves in $\mathcal{C}$ are holomorphic.

The above remarks apply to the general relative inflation process. In fact in our case $H_2(X_k, \mathcal{C})$ is generated by the classes E_3 and E_k that project to the divisors D_1, D_2, and we have already arranged that the forms $\Phi_J^*(\omega_t)$ have the same integral over E_3. Hence we only need to worry about the size of E_k. If this is too big, it is easy to decrease it, essentially by inflating along the representative C_k of E_k itself. (As pointed out by Li–Usher [14], one can also interpret inflation as a symplectic connect sum, and hence can inflate along curves of negative self intersection.) If it is too small and $\kappa \leq 0$ we can increase it to κ by inflating along a curve in the class A_k where

$$A_7 := 5(L - E_{123}) - 6E_{4\ldots k}, \quad A_8 := 11(L - E_{123}) - 12E_{4\ldots k}.$$

Then

$$A_k \cdot A_k > 0, \quad d(A_7) = \tfrac{1}{2}(A \cdot A + c_1(A)) = 6, \quad d(A_8) = 6.$$

These inequalities imply that the Gromov–Witten invariant $Gr(A_k)$ that counts embedded holomorphic curves through $d(A_k)$ points is nonzero, so that these classes have

smooth J-representatives for generic J. Moreover, because $PD(A_k)$ is a multiple of $[\omega_\mu]$ for some $\mu > 0$, A_k can be used to change the cohomology class of ω_κ by increasing κ to any number $< \mu$. In particular, we can increase κ to 0.   □

Proposition 2.20. *Any two cohomologous symplectic forms on Z are diffeomorphic.*

Proof. The argument below basically shows that any symplectic form ω on Z is the blow up of a form ρ on $\mathbb{C}P^2$, so that the result follows from the uniqueness of symplectic structures on $\mathbb{C}P^2$. However, it is easiest to explain the details if we start with two symplectic forms ω', ω'' on Z. We assume as we may that these agree on the neighborhood $\mathcal{N}$ of the singular points $\mathbf{p}$.

We shall work on the resolution X_k. Denote by Ω'_ε and Ω''_ε the symplectic forms on X_k obtained from $\Phi^*_J(\omega')$ and $\Phi^*_J(\omega'')$ by changing them in $\mathcal{V}$ to equal τ_ε. We will show that Ω'_ε is diffeomorphic to Ω''_ε, by a diffeomorphism ϕ that equals the identity in a neighborhood $\mathcal{V}_1 \subset \mathcal{V}$ of $\mathcal{C}$. Since we may recover ω', ω'' from $\Omega'_\varepsilon, \Omega''_\varepsilon$ by the same symplectic blow down process near $\mathcal{C}$, we may extend this diffeomorphism by the identity to get the desired diffeomorphism of Z.

To construct this diffeomorphism of X_k, choose $J' \in \mathcal{J}_{\mathcal{V},reg}(\Omega'_\varepsilon)$, and consider the corresponding J'-holomorphic spheres C'_3 and C'_k in classes E_3 and E_k (which exist by Lemma 2.18.) Our first aim is to extend the local toric model to include these curves C'_3, C'_k. By positivity of intersections, these must each intersect $\mathcal{C}$ transversally. In fact, because $C'_3 \cdot C_0 = 1$, C'_3 meets C_0 transversally at a single point q'_0. Similarly, C'_3 meets C_2 at q'_2, C'_k meets C_{k-1} at q'_k and there are no other intersections. Let q_2, q_0, q_k denote the corresponding points of intersection of $C_3 \cup C_k$ with $\mathcal{C}$. We now claim that there is an Ω'_ε-symplectic isotopy $g_t, t \in [0, 1]$, supported near $\mathcal{C}$ such that $g_0 = id$, $g_t(\mathcal{C}) = \mathcal{C}$ for all t, and so that $g_1(C'_3)$ and $g_1(C'_k)$ coincide with C_3 and C_k near $\mathcal{C}$. To achieve this, we first isotop C'_3 and C'_k so that they meet $\mathcal{C}$ at the correct points, then straighten them out so that the intersection is orthogonal,[7] and finally isotop them to coincide with C_3, C_k near the points q_i.

Therefore we may assume that the J'-holomorphic spheres C'_3 and C'_k are such that $C'_3 = C_3$ and $C'_k = C_k$ near their intersections with $\mathcal{C}$. We denote by $\mathcal{S}'$ the set of symplectic forms on X_k that equal some toric form near $\mathcal{C} \cup C'_3 \cup C'_k$. Similarly, we may choose $J'' \in \mathcal{J}_{\mathcal{V},reg}(\Omega''_\varepsilon)$ such that the corresponding spheres C''_3, C''_k equal C_3, C_k near $\mathcal{C}$. For the other i we define $C'_i := C_i =: C''_i$.

Now consider the set of curves $\cup^k_{i=1} C'_i$, i.e. all the curves except for C_0 in class $L - E_{123}$. This set has two connected components $\mathcal{C}'_1 := \cup^3_{i=1} C'_i$ and $\mathcal{C}'_2 := \cup^k_{i=4} C'_i$ both with local toric models. The symplectic blow down process as described in Symington [28] removes this set of curves, inserting their stead two closed regions $\mathcal{R}_1, \mathcal{R}_2$ whose boundaries collapse to $\mathcal{C}'_1, \mathcal{C}'_2$ under the characteristic flow; see Figure 2.4. Denote this symplectic blow down by (Y', ω'_Y). It contains (embedded copies of) the regions $\mathcal{R}_j, j = 1, 2$, and one can get back to X_k by cutting out the interior of these regions and collapsing their boundaries.

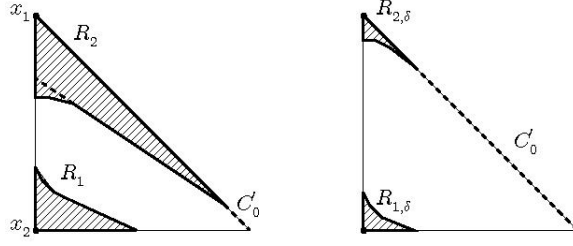[7] This straightening technique is well known: see for example the proof of Theorem 9.4.7 (ii) in [19].

FIGURE 2.4. The regions $\mathcal{R}_j$ that we blow down to get Y, in their original form on the left and made small on the right.

Let x'_1, x'_2 be the points in Y' corresponding to the vertices x_1, x_2 of the simplex in Figure 2.4. Notice that by varying the size of the curves $C'_i, i \geq 1$, (i.e. by moving Ω'_ε along a path in $\mathcal{S}'$) we can make the regions $\mathcal{R}_j$ arbitrarily small. (This is not a question of making ε smaller, since that decreases C_0, but rather of decreasing the size of all curves in $\mathcal{C}'_1 \cup \mathcal{C}'_2$, increasing C_0 correspondingly.) Thus given any neighborhoods of x'_1, x'_2 in Y' we can construct a symplectic form Ω'_δ on X_k that lies in $\mathcal{S}'$ by removing suitably small copies $\mathcal{R}_{j,\delta}$ of $\mathcal{R}_j, j = 1, 2$ in these neighborhoods. Clearly, such a form Ω'_δ can be deformed to the original form Ω'_ε within $\mathcal{S}'$.

Next observe that $H_2(Y') = \mathbb{Z}$, with generator represented by a symplectically embedded 2-sphere S'_0 through q', the image of C'_0. Therefore by Gromov's well known theorem (cf. [19, Ch. 9.4]), (Y', S', ω'_Y) is symplectomorphic to $(\mathbb{C}P^2, \mathbb{C}P^1)$ with its standard form. Applying a similar argument to the curves $\cup_{i=1}^k C''_i$, we get another copy (Y'', ω''_Y) of $\mathbb{C}P^2$, that also contains embedded copies of $\mathcal{R}_j, j = 1, 2$. It follows that there is a symplectomorphism

$$\psi : (Y', S'_0, x'_1, x'_2, \omega'_Y) \;\; \rightarrow \;\; (Y'', S''_0, x''_1, x''_2, \omega''_Y).$$

We can isotop ψ so that, for sufficiently small δ and for $j = 1, 2$, it takes the copy of $\mathcal{R}_{j,\delta}$ in Y' to that in Y''. Then ψ lifts to a symplectomorphism $(X_k, \Omega'_\delta) \to (X_k, \Omega''_\delta)$ on the blow up. Moreover, it can be chosen to be the identity near $\mathcal{C}$ since the curves C'_3, C''_3 and C'_k, C''_k coincide near $\mathcal{C}$. Hence Ω'_ε is deformation equivalent to $\psi^*(\Omega''_\varepsilon)$ by a deformation in $\mathcal{S}'$. One now changes this deformation to an isotopy in $\mathcal{S}'$ by the inflation procedure as described in Lemma 2.19. $\qquad\square$

Remark 2.21. If in the above proof we only had to deal with the singularity at $p_\ell \in Z$ (which is resolved by the curves in $\mathcal{C}'_1$) then we could perform an orbifold blow down directly from Z, with no need to pass to X_k. However, the other two singularities at p_2, p_3 do not have such a direct blow down, and we must first blow up to X_k before passing to the blow down.

To prove that Z satisfies condition (c), we first consider the case of the reduced space Z at level $\kappa \in (-\ell, 2(3 - \ell))$. We saw at the beginning of §3 that in this case (Z, ω_κ) has a toric structure with moment polytope as pictured in Fig 2.2.

Lemma 2.22. If (Z, ω_κ) is toric then the group $\mathrm{Symp}_H(Z, \omega_\kappa)$ is connected.

Proof. Choose $\lambda = 6(\ell + \kappa)/\ell(6 + \kappa)$, the ratio of the integral of ω_κ over D_2 to its integral over D_1; cf. equation (2.7). Since (Z, ω_κ) is toric, the local toric form $\tau_{\lambda,\varepsilon}$ extends to a global toric form Ω on X_k which we may assume to equal a multiple of ω_κ outside $\mathcal{V}$.

Suppose that $\phi \in \mathrm{Symp}(Z, \omega_\kappa)$. Since ϕ acts trivially on homology, parts (ii) and (iii) of Lemma 2.6 imply that ϕ is symplectically isotopic to a symplectomorphism ϕ_1 that is the identity near the singular points $\mathbf{p}$.

Step 1: ϕ_1 *is symplectically isotopic to a symplectomorphism ϕ_2 that is the identity near the divisors D_1 and D_2.*

Note that any symplectomorphism ϕ of Z that is the identity near the singular points $\mathbf{p}$ lifts to a symplectomorphism $\widetilde{\phi}$ of (X_k, τ_ε) that is the identity on some neighborhood $\mathcal{V}$ of $\mathcal{C}$; cf. Figure 2.1.

By Lemma 2.18, there is a path $J_t \in \mathcal{J}_{\mathcal{V},reg}(\Omega)$ from $J_\mathcal{V}$ to $J_1 := (\widetilde{\phi}_1)_*(J_\mathcal{V})$. Let $C_{3,t}, C_{k,t}$ be the J_t-holomorphic representatives of the classes E_3, E_k. They are symplectically embedded by construction, and as in the previous proof we may alter them by a symplectic isotopy supported near $\mathcal{C}$ to families of curves that coincide with $C_3 = C_{3,0}$ and $C_k = C_{k,0}$ near $\mathcal{C}$ for all t. Then, the symplectic isotopy extension theorem implies that there is a family of symplectomorphisms g_t with $g_0 = id$, and such that, for all t,

$$g_t = id \text{ near } \mathcal{C}, \quad g_t(C_3) = C_{3,t}, \quad g_t(C_k) = C_{k,t}.$$

Then $\widetilde{\phi}_2 := g_1^{-1} \circ \widetilde{\phi}_1$ takes C_3 to itself and C_k to itself, and is the identity near the points of intersection with $\mathcal{C}$.

Since C_k intersects $\mathcal{C}$ in a single point, it is easy to adjust the isotopy g_t so that $\widetilde{\phi}_2 = id$ on C_k. However, C_3 meets $\mathcal{C}$ in two points and so the induced map on $C_3 \smallsetminus \mathcal{C}$ may not be isotopic to the identity by an isotopy of compact support, although there is an isotopy to the identity that fixes C_3 near the point $C_3 \cap C_2$ and rotates C_3 near $C_3 \cap C_0$. On the other hand, it is not essential to consider only those symplectomorphisms that are the identity on $\mathcal{C}$ since all we need is that the symplectomorphisms on X_k descend to symplectomorphisms on Z. Therefore we just need to check that there is an S^1 action near the singular point p_2 in Z that lifts to an action near C_0 that fixes the point $C_0 \cap C_3$ but rotates both C_0 and a neighborhood of the point $C_0 \cap C_3$ in C_3. But this is clear because our local models are toric and there is a suitable S^1 subgroup of the torus T^2.

Hence we may assume that $\widetilde{\phi}_2$ is the identity on $C_3 \cup C_k$ as well as near $\mathcal{C}$ and then make a final isotopy in the directions normal to $C_3 \cup C_k$ to make it the identity on a neighborhood. This gives the desired isotopy in X_k. Since all the symplectomorphisms considered are either equal to the identity near $\mathcal{C}$ or equal to a rotation that is contained in the local torus actions, they push forward to Z, yielding the desired isotopy of ϕ_1 to ϕ_2.

Step 2: *We may isotop ϕ_2 to a symplectomorphism ϕ_3 that is also the identity near the divisor D_3 represented by the edge $v_1 v_2$ in the toric model of Figure 2.5.*

This edge pulls back to a curve C' in X_k in the class $L - E_{14}$. This class is again in $\mathcal{E}(X_k)$ and so has a smooth representative for $J \in \mathcal{J}_{\mathcal{V},reg}(\Omega)$. Therefore this step may be accomplished by arguing as in Step 1. Note that again C' intersects $\mathcal{C}$ in two points. Therefore to make $\phi_3 = id$ on D_3 we may need to rotate Z near its singular point of order k. This is possible as before.
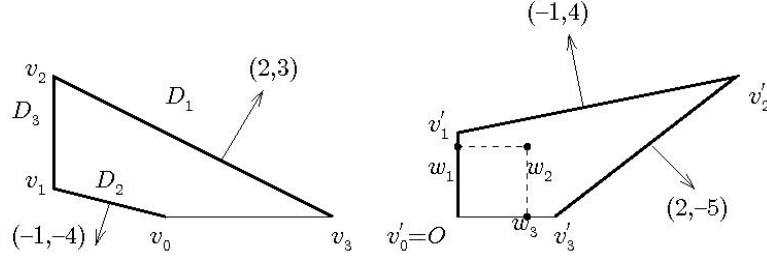

FIGURE 2.5. The original moment polytope is on the left and its affine image is on the right. The three divisors D_1, D_2, D_3 are represented by the thicker lines.

Step 3: *Shrinking the support of ϕ_3.*

We have now isotoped ϕ to a symplectomorphism ϕ_3 that is the identity near the divisors $D_i, i = 1, 2, 3$, represented by the three edges v_2v_3, v_0v_1 and v_1v_2 of the moment polytope Δ. Because v_0 is a smooth point of the polytope, we may by an affine change of coordinates identify Δ with the polytope $O = v_0', v_1', v_2', v_3'$ where O, the image of the point v_0, is at the origin; cf. Figure 2.5. It follows that the open set $\mathcal{U}_Z = Z \smallsetminus (D_3 \cup D_1)$, the inverse image under the moment map of $\Delta \smallsetminus (v_1v_2 \cup v_2v_3)$, has a natural Darboux chart whose image $\mathcal{U}_0$ is the open convex subset of $(\mathbb{C}^2, \omega_0)$ defined by the two equations

$$-|z_1|^2 + 4|z_2|^2 < c_1, \quad 2|z_1|^2 - 5|z_2|^2 < c_2,$$

for some $c_i > 0$. (Note that the coefficients in these equations are given by the conormals to the edges $v_1'v_2', v_2'v_3'$.) Moreover, in these coordinates, the divisor D_2 corresponds to the disc $z_1 = 0$.

For $0 < \lambda < 1$ let m_λ be the image in $\mathcal{U}_Z$ of the rescaling map $\mathcal{U}_0 \to \mathcal{U}_0$ given by multiplication by λ. Since ϕ_3 has support in $\mathcal{U}_Z$ the symplectomorphism

$$\phi_{3+t} := m_{1-t} \circ \phi_3 \circ m_{1-t}^{-1}$$

is well defined for all $t \in [0, 1)$ and has support in $m_{1-t}(\mathcal{U}_Z)$. In particular it is the identity on D_2 for all t. Moreover, for t sufficiently close to 1 its support maps into a square of the form $Ow_1w_2w_3$. Thus its support is contained in the interior of a subset of (Z, ω_κ) symplectomorphic to the product $P := (S^1 \times [0, 1]) \times D^2$ with a product symplectic form ω_0 that has the same integral over the two factors.

Now denote by $\mathrm{Symp}(P, \partial P; \omega_0)$ the group of symplectomorphisms of (P, ω_0) that are the identity near the boundary. Since the first three steps isotop ϕ to an element of $\mathrm{Symp}(P, \partial P; \omega_0)$, the following step completes the proof.

Step 4: $\mathrm{Symp}_0(P, \partial P; \omega_0)$ *is contractible.*

We may identify this group with the subgroup

$$\mathcal{G}_0 = \left\{ g \in \mathrm{Symp}(S^2 \times S^2, \sigma \times \sigma) \,|\, g = id. \text{ near } S^2 \times \{0\} \cup \{0, \infty\} \times S^2 \right\},$$

of the group $\mathcal{G}$ of symplectomorphisms of $(S^2 \times S^2, \sigma \times \sigma)$ that are the identity on $(\{0\} \times S^2) \cup (S^2 \times \{0\})$. It follows from work of Gromov that $\mathcal{G}$ is contractible; see the survey article [11] or [19, Ch 9.5]. Moreover, there is a fibration sequence

$$\mathcal{G}_0 \to \mathcal{G} \xrightarrow{ev} \mathcal{E}mb$$

where $\mathcal{E}mb$ is the space of symplectic embeddings $\bar{g} : \{\infty\} \times S^2 \to S^2 \times S^2$ that extend to elements of $\mathcal{G}$. (Notice that the fiber of ev consists in fact of maps that are the identity *on* $\{\infty\} \times S^2$ and *near* the point $\{\infty\} \times \{0\}$, but not *near* the whole of this sphere. But a standard Moser argument shows that the space of such maps is homotopy equivalent to $\mathcal{G}_0$.) Because the two 2-spheres in $S^2 \times S^2$ have the same size, it follows as in [11, 19] that $\mathcal{E}mb$ is homotopy equivalent to the contractible space of $(\sigma \times \sigma)$-tame almost complex structures on $S^2 \times S^2$ that equal the product structure near $S^2 \times \{0\} \cup \{0, \infty\} \times S^2$. Therefore $\mathcal{E}mb$ and hence also $\mathcal{G}_0$ is contractible. $\qquad\square$

Lemma 2.23. $\mathrm{Symp}_H(Z, \omega_\kappa)$ *is connected for all* $\kappa \in (-\ell, 0]$.

Proof. We shall suppose that κ is too large for (Z, ω_κ) to be toric, since otherwise there is nothing to prove. Denote by $\mathrm{Diff}_0^c(Z \smallsetminus \mathbf{p})$ the identity component of the group of compactly supported diffeomorphisms of $Z \smallsetminus \mathbf{p}$. We give this (and all similar spaces) the usual direct limit topology so that the elements in any compact subset of $\mathrm{Diff}_0^c(Z \smallsetminus \mathbf{p})$ all equal the identity on some fixed neighborhood of $\mathbf{p}$. Similarly, let $\mathrm{Symp}^c(Z \smallsetminus \mathbf{p}, \omega_\kappa)$ be the subgroup of compactly supported elements of $\mathrm{Symp}(Z \smallsetminus \mathbf{p}, \omega_\kappa)$. Elements of this group must fix the classes $e(Y), [\omega_\kappa]$ by Lemma 2.6 and so act trivially on homology.

We will assume as we may that ω_κ is standard in some neighborhood of $\mathbf{p}$, and denote by Ω the symplectic form on X_k that equals τ_ε on $\mathcal{V} := \Phi_J^{-1}(\mathcal{N})$ and equals $\Phi_J^*(\omega_\kappa)$ on $X_k \smallsetminus \mathcal{V}$.

Step 1. *It suffices to show that* $\mathrm{Symp}^c(Z \smallsetminus \mathbf{p}, \omega_\kappa)$ *is path connected.*

As in the second paragraph of the proof of Lemma 2.22, every element in $\mathrm{Symp}_H(Z, \omega_\kappa)$ is isotopic to a symplectomorphism that is the identity near $\mathbf{p}$.

Step 2. $\mathrm{Symp}^c(Z \smallsetminus \mathbf{p}, \omega_\kappa) \subset \mathrm{Diff}_0(Z, \mathbf{p})$.

Every $\phi \in \mathrm{Symp}^c(Z \smallsetminus \mathbf{p}, \omega_\kappa)$ lifts to a symplectomorphism $\widetilde{\phi}$ of (X_k, Ω) that is the identity in $\mathcal{V}$. As in Step 1 of the proof of Lemma 2.22, there is a path $\widetilde{\phi}_t \in \mathrm{Symp}(X_k, \mathcal{C}, \Omega)$ starting at $\widetilde{\phi}$ and ending at an element $\widetilde{\phi}_1$ that is the identity in some neighborhood $\mathcal{N}(C_k)$ of C_k. Then change the symplectic form Ω in $\mathcal{N}(C_k)$, decreasing the size of C_k, to a form Ω' that lies in a class with a toric representative. Then $\widetilde{\phi}_1$ preserves the form Ω' and so is the lift of an element ϕ_1 in $\mathrm{Symp}(Z, \omega_{\kappa'})$, where $\omega_{\kappa'}$ is homologous to a toric form. Therefore $\omega_{\kappa'}$ is diffeomorphic to a toric form by Proposition 2.20 and we can apply Lemma 2.22 to conclude that ϕ_2 is smoothly isotopic to the identity. Since ϕ is smoothly isotopic to ϕ_1 by construction, this proves Step 1.

Step 3. $\operatorname{Symp}^c(Z \diagdown \mathbf{p}, \omega_\kappa)$ *is path connected.*

Denote by $\mathcal{SF}(\kappa)$ the space of all symplectic forms that are isotopic to ω_κ through a family $\omega_{t,\kappa}, t \in [0,1]$, of symplectic forms that are all standard in some fixed neighborhood of $\mathbf{p}$. Since $\operatorname{Diff}_0^c(Z \diagdown \mathbf{p})$ acts transitively on $\mathcal{SF}(\kappa)$, there is a fibration sequence

$$\operatorname{Symp}^c(Z \diagdown \mathbf{p}, \omega_\kappa) \;\to\; \operatorname{Diff}_0^c(Z \diagdown \mathbf{p}) \;\stackrel{\alpha_\kappa}{\to}\; \mathcal{SF}(\kappa).$$

It suffices to show that the map $(\alpha_\kappa)_* : \pi_1(\operatorname{Diff}_0(Z, \mathbf{p})) \to \pi_1(\mathcal{SF}(\kappa))$ is surjective. By Lemma 2.22 there is $\kappa' < \kappa$ for which this holds. Therefore it suffices to construct a map $r : \mathcal{SF}(\kappa) \to \mathcal{SF}(\kappa')$ such that $\alpha_{\kappa'}$ is weakly homotopic to $r \circ \alpha_\kappa$.

To this end, we use some ideas from [16]. (As explained at the end of the proof, this approach gives somewhat more than we need.) Denote by $J_\mathcal{N}$ the image of $J_\mathcal{V}$ under the blow down map $X_k \to Z$. Consider the space $\mathcal{A}(\kappa)$ of all almost complex structures J on Z that are tamed by some form in $\mathcal{SF}(\kappa)$ and are equal to $J_\mathcal{N}$ near $\mathbf{p}$. Further, define $\mathcal{X}(\kappa)$ to be the space of all pairs $(\omega, J) \in \mathcal{SF}(\kappa) \times \mathcal{A}(\kappa)$ such that ω tames J. Then the projection map $\mathcal{X}(\kappa) \to \mathcal{SF}(\kappa)$ has contractible fibers and so is a homotopy equivalence. A similar statement holds for the projection $\mathcal{X}(\kappa) \to \mathcal{A}(\kappa)$. (Because everything is normalized near $\mathbf{p}$, the singular points cause no problem.)

We now claim that $\mathcal{A}(\kappa) = \mathcal{A}(\kappa')$ for all $\kappa' < \kappa$. This holds by Lemma 2.18 (ii). For every $J \in \mathcal{A}(\kappa)$ there is a unique embedded J-holomorphic curve C_J in class E_k. If J is tamed by $\omega \in \mathcal{SF}(\kappa)$, ω is nondegenerate on C_J and therefore we can inflate ω along C_J, constructing a family of forms $\omega_\lambda, \kappa' \leq \lambda \leq \kappa$, that

- tame J,
- equal ω away from C_k and
- are such that $\int_{C_k} \omega_\lambda = \lambda$.

For details, see [16]. (The argument needed for this is a little more delicate than in the usual inflation procedure since the forms ω_λ must tame J.)

This argument shows that the spaces $\mathcal{SF}(\kappa)$ and $\mathcal{SF}(\kappa')$ are homotopy equivalent. Further, we can define a map $r : \mathcal{SF}(\kappa) \to \mathcal{SF}(\kappa')$ that induces this equivalence and is unique up to homotopy, as follows: given a compact family $\mathcal{M} = \{\omega_\mu\}$ of elements of $\mathcal{SF}(\kappa)$ choose a corresponding family J_μ of ω_μ tame almost complex structures, and then alter the ω_μ appropriately near the curves C_{J_μ} to a family $\omega_{\mu,\lambda}$. There are choices here, but they are equivalent up to homotopy.

Note finally that if all we aim to do is construct this map r we can use the less delicate version of inflation: there is no need to insist that the modified forms $\omega_{\mu,\lambda}$ are J_μ tame. Also, if we are only interested in π_1 we can restrict to one dimensional families $\mathcal{M}$.

Since $\alpha_{\kappa'}$ is clearly weakly homotopic to $r \circ \alpha_\kappa$, this completes the proof. $\qquad \square$

Remark 2.24. The argument in Step 3 above shows that the homotopy type of the group $\operatorname{Symp}_0^c(Z \diagdown \mathbf{p}, \omega_\kappa)$ is independent of $\kappa \in (-\ell, 0]$. In contrast, the homotopy type of the groups $\operatorname{Symp}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ vary with the cohomology class of the form ω_λ. However, the one point blow up of $\mathbb{C}P^2$ is the unique manifold for which Pinsonnault's result quoted in Lemma 2.18 (ii) fails to hold.

Remark 2.25. Although we have carried out the proof of Theorem 2.16 for the orbifold Z, most of the arguments apply much more widely. For example, the uniqueness statement of Proposition 2.20 easily extends to the case when Z is an orbifold blow up of the weighted projective space $\mathbb{C}P^2(1, a, b)$ at k distinct points, provided that a, b are relatively prime. (If a, b are not relatively prime then the singularities of $\mathbb{C}P^2(1, a, b)$ are no longer isolated points, and one would have to use a different kind of resolution.) One might even be able to extend it further (for example to blowups of any $\mathbb{C}P^2(a, b, c)$), perhaps by using the techniques developed to understand fillings of simple singularities as in Ohta–Ono [23]. Chen has a different approach to these questions that is based on extending Seiberg–Witten–Taubes theory to the orbifold setting; cf. Chen [2].

Similarly the *deformation implies isotopy* property of Z is very general, and should hold for any blow up that is resolved by some N-fold blow up of $\mathbb{C}P^2$. However, the *connectness* property is more delicate, just as it is in the case of the X_N.

3. CONSTRUCTION AND PROPERTIES OF M_ℓ.

Most of the first subsection is devoted to the existence proof. However it also contains Lemma 3.2 which, together with Lemma 3.4 in §3.2, are the basic ingredients of the uniqueness proof. The last subsection §3.3 discusses the cases $\ell = 2, 3$.

3.1. **Existence.** We first prove the following result.

Proposition 3.1. *For each ℓ, there is a Hamiltonian S^1 manifold $(M^{\leq 0}, \Omega^-)$ with boundary (Y^-, Ω^-) whose reduced spaces at level κ are*

$$\begin{cases} \left(\mathbb{C}P^2_{1,2,3}, \frac{6+\kappa}{6}\tau_{1,2,3}\right) & when \ -6 \leq \kappa < -\ell, \\ (Z, \omega_\kappa) & when \ -\ell < \kappa \leq 0. \end{cases}$$

Its proof is based on the following well known lemma. In it, the word "unique" means unique up to equivariant symplectomorphism.

Lemma 3.2. *Suppose that (M, Ω) is a (possibly noncompact) Hamiltonian S^1-bundle with proper moment map $H : M \to \mathbb{R}$. Let $I \subset \mathbb{R}$ be an interval that contains no critical points of H with corresponding family of reduced spaces (V, ω_κ). Then the slice $H^{-1}(I)$ is uniquely determined by the family of forms $\omega_\kappa, \kappa \in I$, on the orbifold V.*

Moreover, given any family of symplectic forms $\omega_\kappa, \kappa \in I$, on an orbifold V such that $\frac{d}{d\kappa}\omega_\kappa$ is an integral class whose corresponding orbibundle $S^1 \to Y \xrightarrow{\pi} V$ has smooth total space, there is a (unique) corresponding Hamiltonian S^1 manifold (M, Ω) with reduced spaces (V, ω_κ).

Sketch of proof. All statements here, except possibly for the uniqueness, are well known. If the action on $H^{-1}(I)$ is free, then this is proved in [18, Prop 5.8]. The key point is to write Ω on $M \equiv Y \times I$ in the form

$$\Omega = \pi^*(\omega_\kappa) + \alpha_\kappa \wedge d\kappa,$$

where α_κ is a suitable family of connection 1-forms on the circle bundle $\pi : Y \to V$. The argument in the general case is similar; one simply has to understand the behavior

of connection 1-forms on S^1-orbibundles. For further details, see Karshon–Tolman [9, §3]. □

Remark 3.3. In [9] Karshon and Tolman work with "complexity one" spaces, i.e. manifolds of dimension $2k \geq 4$ with Hamiltonian actions of T^{k-1}. In this case, the reduced spaces V_κ have real dimension 2 so that their symplectic structure is determined by cohomological information — indeed, just by the Duistermatt–Heckmann measure. The arguments and definitions in [9, §3] carry over to the case when the reduced spaces are rigid in the sense of Definition 2.13. For example, Gonzalez shows in [7] that in the rigid case the total space (M, Ω) depends only on the cohomology classes $[\omega_\kappa], \kappa \in I$. However, in the general case considered in Lemma 3.2, the family of forms may contain some more information. Therefore, to use the language of [9] one needs to formulate an appropriate redefinition of Karshon–Tolman's concept of a $\Phi - T$-diffeomorphism: besides commuting with the moment map, the induced family of diffeomorphisms on the reduced space V should preserve the family of forms ω_κ.

Proof of Proposition 3.1. As illustrated in Figure 3.1 we may construct the sublevel set $(M^{\leq -\ell + \varepsilon}, \Omega^-)$ as a toric manifold. When translated vertically upwards by 6 so that its lowest vertex is at the origin and $x_3 = \kappa + 6$, its moment polytope in $\mathbb{R}^3$ is described by the inequalities:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad 2x_1 + 3x_2 \leq \tfrac{x_3}{6},$$
$$x_1 + 4x_2 \geq 1 - \tfrac{6}{\ell} + \tfrac{x_3}{\ell}, \quad 0 \leq x_3 \leq 6 - \ell + \varepsilon.$$

The slice $(M^{(-\ell, -\ell + \varepsilon)}, \Omega)$ is a union of circle (orbi-)bundles over the reduced spaces (Z, ω_κ) and, by Lemma 3.2, may be extended by attaching circle (orbi)bundles over (Z, ω_κ) for $-\ell < \kappa \leq 0$. □
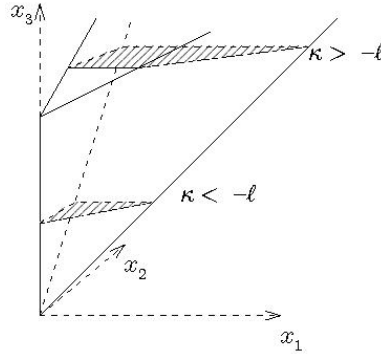

FIGURE 3.1. The moment polytope for $M^{\leq -\ell + \varepsilon}$ showing its two types of critical level. Here $x_3 = \kappa + 6 \geq 0$.

The manifold $(M^{\leq 0}, \Omega^-)$ realises the sublevel set $H^{-1}([-6, 0])$. Denote by $(M^{\geq 0}, \Omega^+)$ the Hamiltonian S^1 manifold that is diffeomorphic to $(M^{\leq 0}, \Omega^-)$ but has the reversed

S^1 action. In other words, if ι denotes the identity map and $-id : S^1 \to S^1$ takes t to $-t$, then there is a commutative diagram

$$
\begin{array}{ccc}
S^1 \times M^{\leq 0} & \xrightarrow{-id \times \iota} & S^1 \times M^{\geq 0} \\
\alpha^- \downarrow & & \alpha^+ \downarrow \\
M^{\leq 0} & \xrightarrow{\iota} & M^{\geq 0},
\end{array}
$$

where $\alpha^\pm$ is the action on $M^{\gtrless 0}$. Then the Hamiltonian generating α^+ is $-H \circ \iota$, and ι induces a symplectomorphism between the reduced space of $M^{\leq 0}$ at level $\kappa \leq 0$ and that of $M^{\geq 0}$ at level $-\kappa \geq 0$.

Proof of Lemma 1.2. This lemma states that to construct (M_ℓ, Ω) it suffices to find a symplectomorphism of (Z, ω_0) that changes the sign of the Euler class $e(Y)$. Thus, from such a symplectomorphism ϕ_Z we need to find a way to glue $(M^{\leq 0}, \Omega^-)$ to $(M^{\geq 0}, \Omega^+)$ along their common boundary in an equivariant and smooth way. We shall prove this by applying Lemma 3.2.

In the following argument $\delta > 0$ is a small constant that may be decreased as needed. First choose a smooth family $\psi_\kappa, \kappa \in (-\delta, 0]$, of diffeomorphisms from the quotient spaces $H^{-1}(\kappa)/S^1$ to Z. Let σ_κ, $\kappa \in (-\delta, 0]$, be the corresponding smooth family of symplectic forms induced by Ω^-. By adjusting ψ_κ we may assume that $\sigma_0 = \omega_0$ and that the σ_κ are standard in some neighborhood $\mathcal{N}$ of the singular points $\mathbf{p}$. Then the Duistermatt–Heckman formula implies that the 2-form

$$
\lambda := \frac{d}{dt} \sigma_\kappa \Big|_{t=0}
$$

represents the class $-e(Y)$, and by further adjusting the ψ_κ (and deceasing δ) we may suppose that

$$
\sigma_\kappa = \omega_0 + \kappa \lambda, \quad \kappa \in (-\delta, 0].
$$

Then Lemma 3.2 implies that a neighborhood of the boundary of $(M^{\leq 0}, \Omega^-)$ is determined by this family $\sigma_\kappa, \kappa \in (-\delta, 0]$.

Now define the 1-form γ on Z by the equation $\phi_Z^*(\lambda) = -\lambda + d\gamma$. Since $H^2(Z) = H^2(Z, \mathcal{N})$, we may suppose that $\gamma = 0$ in $\mathcal{N}$. By a standard Moser argument, there is a smooth family of diffeomorphisms $f_\kappa : Z \to Z$ that are the identity in $\mathcal{N}$ and such that for small δ

$$
f_0 = id, \quad f_\kappa^*(\omega + \kappa \lambda) = \omega_0 + \kappa(\lambda - (\phi_Z^{-1})^* d\gamma), \quad \kappa \in (-\delta, 0].
$$

Now set

$$
\omega_\kappa := \phi_Z^*\big(f_{-\kappa}^*(\omega_{-\kappa})\big) = \omega_0 + \kappa \lambda, \quad \kappa \in [0, \delta).
$$

Again, Lemma 3.2 implies that a neighborhood of the boundary of $(M^{\geq 0}, \Omega^+)$ is determined by this family $\omega_\kappa, \kappa \in [0, \delta)$. But these two families fit smoothly together over $(-\delta, \delta)$. Hence the result follows by another application of Lemma 3.2. $\qquad \square$

Proof of Proposition 1.3 part (i). We must show that there is a symplectomeophism ϕ_Z as above. To see this, consider the reduced space (Z, ω_0) at level zero. The form $\Phi_J^*(\omega_0)$ is cohomologous to $-K = [\tau]$ (cf. equation (1.1)) since it vanishes on the contracted set $\mathcal{C}$ and takes the value 1 on E_3, E_k by Example 2.3. (Note that the form

$\Phi_J^*(\omega_0)$ is degenerate along $\mathcal{C}$ and so is not symplectic.) Further by equations (2.6) and (2.10)

$$(3.1) \qquad \Phi_J^*(e(Y))(C_k) = e(Y)(\mathbb{C}P_{1,\ell}^1) = -\tfrac{1}{\ell}, \quad \Phi_J^*(e(Y))(C_3) = -\tfrac{1}{6},$$

$$\Phi_J^*(e(Y))(L - E_{123}) = e(Y)(\Phi_{J*}(C_0)) = 0.$$

Hence $\chi_k = -\Phi_J^*(e(Y))$.

Now consider the case $\ell = 4$. By Lemma 1.4 there is a diffeomorphism ψ of X_7 that reverses the sign of $\Phi_J^*(e(Y))$. Denote $\mathcal{C}' := \psi(\mathcal{C})$ and $J' := \psi_* J$, and let Z' be the image of (X_7, J') under the map $\Phi_{J'}$ that contracts the curves in $\mathcal{C}'$. Observe that $\mathcal{C}'$ is a union of J'-holomorphic curves C_i' in classes $\widehat{L} - \widehat{E}_{123}, \widehat{E}_1 - \widehat{E}_2$, and so on; that is, the C_i' have the same formulas as do the C_i but with L, E_j replaced by $\widehat{L}, \widehat{E}_j$. Further $\psi : X_7 \to X_7$ descends to a diffeomorphism $\psi_Z : Z \to Z'$. Thus we have the middle part of the diagram

$$(3.2) \qquad \begin{array}{ccc} (X_7, J) & \xrightarrow{\psi} & (X_7, J') \\ \Phi_J \downarrow & & \downarrow \Phi_{J'} \\ \end{array}$$
$$Z^- \xleftarrow{f} Z \xrightarrow{\psi_Z} Z' \xrightarrow{f'} Z^+.$$

We have constructed $(M^{\leq 0}, \Omega^-)$ so that there is a symplectomorphism $f : (Z, \omega_0) \to (Z^-, \omega_0^-)$ such that $(f \circ \Phi_J)^*(e_Z(Y^-)) = \chi_7$. Similarly, it follows from equation (2.4) that if we allow the classes $\widehat{L}, \widehat{E}_j$ (with Poincaré duals $\widehat{a}, \widehat{e}_i$) to play the roles of L, E_j we can construct a diffeomorphism $f' : Z' \to (Z^+, \omega_0^+)$ such that

$$\Phi_{J'}^* (f')^*(e_Z(Y^+)) = \tfrac{1}{12}\big(6\widehat{a} - 2\widehat{e}_{123} - 3\widehat{e}_{4567}\big) = -\chi_7.$$

Denote by $\omega_0' := (f')^*(\omega_0^+)$ the corresponding symplectic form on Z'. The symplectic forms $\psi_Z^*(\omega_0')$ and ω_0 on Z pull back to cohomologous forms on X_7 and hence are themselves cohomologous; cf. the proof of Lemma 2.9. Therefore Proposition 2.20 provides a diffeomorphism $g : Z \to Z$ such that $g^*(\psi^*(\omega_0')) = \omega_0$. Now take ϕ_Z to be the composite:

$$(Z^-, \omega_0^-) \xrightarrow{f^{-1}} (Z, \omega_0) \xrightarrow{\psi_Z \circ g} (Z', \omega_0') \xrightarrow{f'} (Z^+, \omega_0^+).$$

This completes the proof for the case $\ell = 4$. The case $\ell = 5$ is similar. $\qquad \square$

3.2. Uniqueness.

It remains to prove the uniqueness statement. We first prove that the germ of M around a critical level is unique. Then, as in Gonzalez [7], uniqueness will follow from the rigidity of the reduced spaces.

Since this is the only case needed here, we shall suppose that the critical level Y_0 contains a single critical point q with isotropy weights (a_1, a_2, a_3) where $a_1 = -1$, and $a_2, a_3 > 0$. As pointed out by Karshon–Tolman [9], the difficulty is that the critical level Y_0 is not a smooth submanifold near x_0, and so its quotient V_0 by S^1 does not have a natural smooth structure near the image p of q (although V_0 is diffeomorphic to the reduced spaces $V_\kappa, \kappa < 0$, at levels immediately below). We therefore define the smooth structure on V_0 near p by choosing an equivariant Darboux chart for the

smooth manifold M at q modelled on the S^1 space $(\mathbb{C}^3, 0)$ with action and moment map

$$(z_1, z_2, z_3) \mapsto (e^{2\pi i a_1} z_1, e^{2\pi i a_2} z_2, e^{2\pi i a_3} z_3), \quad (z_1, z_2, z_3) \mapsto \sum a_j |z_j|^2.$$

(The symplectic form on $\mathbb{C}^3$ is an appropriate multiple of the standard form.) The equivariant Darboux theorem implies that this chart, a baby version of the "grommets" of [9], is unique up to equivariant isotopy. Moreover, because $a_1 = -1$, the map $\mathbb{C}^2 \to H^{-1}(0) \subset \mathbb{C}^3$ given by

$$(3.3) \qquad\qquad \rho \; : \; (w_2, w_3) \mapsto (\sqrt{a_2 |w_2|^2 + a_3 |w_3|^2}, w_2, w_3)$$

meets each orbit in $H^{-1}(0)$ precisely once and hence provides a coordinate chart for a neighborhood of p in V_0. Putting this together with the natural (quotient) smooth structure on $V_0 \smallsetminus p$ we get a smooth structure on V_0 that is independent of choices. Further, the symplectic form Ω on M descends to a symplectic form ω_0 on V_0 that is again independent of choices.

Lemma 3.4. *Suppose we are given two Hamiltonian S^1 manifolds (M, Ω) and (M', Ω') with proper moment maps H, H', each having an isolated critical point of index $(-1, a_2, a_3)$ at level zero. If the critical reduced levels (V_0, ω_0) and (V_0', ω_0') are symplectomorphic, then for some $\varepsilon > 0$ there is an equivariant symplectomorphism*

$$\Psi : \big(H^{-1}(-\varepsilon, \varepsilon), \Omega \big) \to \big((H')^{-1}(-\varepsilon, \varepsilon), \Omega' \big).$$

Proof. We shall first lift the given symplectomorphism ψ_0 from the critical reduced space V_0 to the critical level Y_0, and then extend this lift Ψ_0 to a symplectomorphism defined on a neighborhood of Y_0 by using a modified gradient flow.

For the first step, choose a Darboux chart $\chi : U \to U_0$ from a neighborhood U of the fixed point $q \in Y_0 \subset M$ to a neighborhood U_0 of 0 in the standard model $\mathbb{C}^3$ described above. Make a similar choice $\chi' : U' \to U_0$ for M'. Isotop the given symplectomorphism $\psi_0 : (V_0, \omega_0) \to (V_0', \omega_0')$ so that it is the identity in the standard coordinates near the critical points p, p'. More precisely, with ρ as in equation (3.3), arrange that

$$(\rho^{-1} \circ \chi') \circ \psi_0 \circ (\chi^{-1} \circ \rho) \, (w_2, w_3) = (w_2, w_3).$$

Then ψ_0 lifts to $(\chi')^{-1} \circ \chi$ in $U \cap Y_0$. Since there are no fixed points except for q, q', and since ψ_0 is a symplectomorphism, one can show as in Lemma 3.2 that this local lift extends to an equivariant map $\Psi_0 : Y_0 \to Y_0'$ such that $\Psi_0^*(\Omega') = \Omega$.

To extend Ψ_0 further, choose an invariant Ω-compatible almost complex structure J on M that equals the standard almost complex structure $\chi^*(J_0)$ on U, and let g be the corresponding metric. Consider the downwards g-gradient flow of H on $H^{-1}([0, \varepsilon))$. If we choose $\varepsilon > 0$ sufficiently small, we may suppose that $U \cap H^{-1}([0, \varepsilon))$ contains all orbits in $H^{-1}([0, \varepsilon))$ whose downward flow converges to q. (These points form the 4-dimensional stable manifold W_S of q and lie above the exceptional divisors in the reduced spaces.) For each $\kappa \in [0, \varepsilon)$ define $\mathcal{F}_\kappa(x)$ to be q, if $x \in W_S$, and otherwise to be the point where the downward gradient flow line through x meets Y_0. Thus $\mathcal{F}_\kappa : (Y_\kappa, W_S) \to (Y_0, q)$ induces a diffeomorphism $Y_\kappa \smallsetminus W_S \to Y_0 \smallsetminus q$. Similarly, if $\kappa < 0$
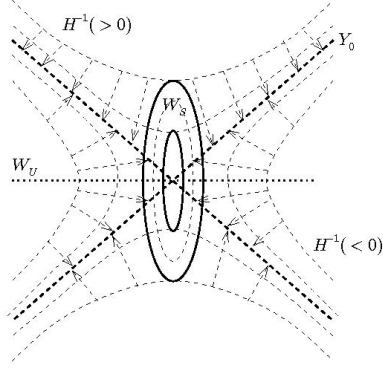
FIGURE 3.2. The maps $\mathcal{F}_\kappa$ in U. The stable manifold W_S is represented by some circles, the critical level Y_0 by a pair of heavy dashed lines, and the unstable manifold W_U by a heavy dotted line.

define $\mathcal{F}_\kappa : (Y_\kappa, W_U) \to (Y_0, q)$ by using the upward gradient flow. (Here W_U is the 2-dimensional unstable manifold of q.)

Define $\mathcal{F}_\kappa'$ similarly on M', and then consider the map Ψ that is defined near Y_0 by

$$\Psi(x) := \begin{cases} (\mathcal{F}_\kappa')^{-1} \circ \Psi_0 \circ \mathcal{F}_\kappa(x), & \text{if } x \in H^{-1}(\kappa) \setminus U \text{ for some } |\kappa| < \varepsilon, \\ (\chi')^{-1} \circ \chi, & \text{if } x \in U. \end{cases}$$

It is easy to check that Ψ is smooth and equivariant. Moreover, it is a symplectomorphism in U and preserves the symplectic form on Y_0. Hence a standard Moser argument shows that it can be equivariantly isotoped, by an isotopy that is the identity near q, to an equivariant symplectomorphism defined near Y_0. $\square$

Proof of Proposition 1.3 part (ii). It follows from Theorem 2.16 that the gluing map ϕ_Z in diagram (3.2) is unique up to symplectic isotopy. Hence (M, Ω) will be unique (up to equivariant symplectomorphism) provided that the sublevel set $(M^{\leq 0}, \Omega^-)$ is. By Lemma 3.7 in [7], the rigidity of the reduced levels of M implies that, for $I = [-6, -\ell)$ and $I = (-\ell, 0]$, any two families $\omega_\kappa, \omega_\kappa'$, $\kappa \in I$, of symplectic forms with $[\omega_\kappa] = [\omega_\kappa']$ for all κ are isotopic through such families. Hence, Lemma 3.2 imply that the slices $M^{[-6,-\ell)}$ and $M^{(-\ell,0]}$ have a unique structure. But the germ of M around the critical level $\kappa = -\ell$ is unique by Lemma 3.4. Therefore the result follows because the maps that glue these pieces together are also unique up isotopy. $\square$

3.3. **The cases $\ell = 2, 3$.** When $\ell = 2, 3$ there is a similar resolution $\Phi_J : X_k \to Z, k = \ell + 3$. When $k = 5$ we may assume that the automorphism $\psi : (X_k, \omega) \to (X_k, \omega)$ takes L, E_i to

$$\begin{aligned} \widehat{L} &= 3L - E_{1\ldots 4} - 2E_5, \\ \widehat{E}_i &= L - E_{j_i 5}, \quad \text{where } j_i = 4 - i \text{ for } 1 \leq i \leq 3, \text{ and } j_5 = 4; \\ \widehat{E}_4 &= 2L - E_{1\ldots 5}. \end{aligned}$$

Moreover, as in equation (3.1), $\Phi_J^*([\omega_\kappa])$ must vanish on $L - E_{123}$ while it is determined on E_3 and E_5 by equation (2.7). Hence

$$\Phi_J^*([\omega_\kappa]) \;\; = \;\; (1 + \tfrac{\kappa}{6})\big(3a - e_1 - e_2 - e_3\big) - (1 + \tfrac{\kappa}{2})(e_4 + e_5).$$
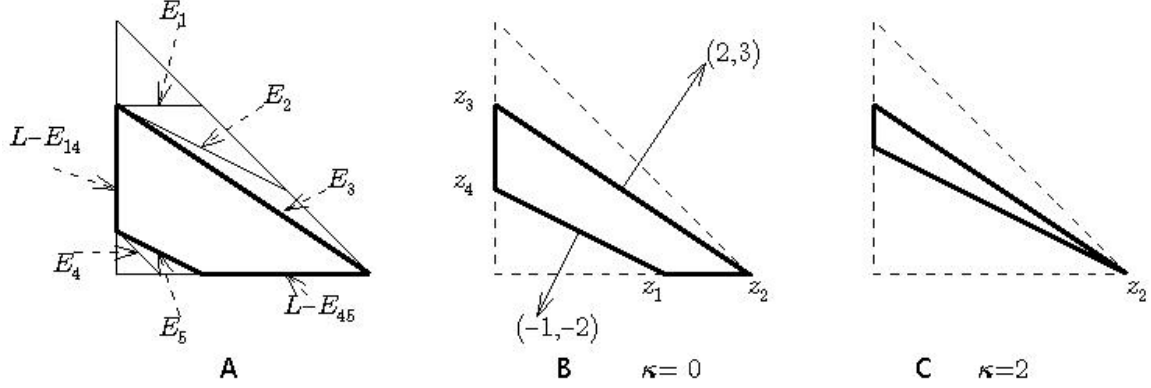

FIGURE 3.3. These are diagrams of the moment polytope of the reduced spaces (Z, ω_κ) for different κ in the case $\ell = 2$. (A) is a schematic representation of Z as a blow up for κ just larger than -2. (cf. [17]); (B) shows the level $\kappa = 0$ in which z_1 is smooth, and z_2, z_4 have order 2; (C) shows the critical level $\kappa = 2$ with z_2 smooth.

We have chosen these formulas as in Lemma 4.2 so that, in the notation introduced there, $\mathcal{C}_0 = \mathcal{C}_0'$ and the curves in classes $\widehat{E}_3 = L - E_{14}$ and $\widehat{E}_5 = L - E_{45}$ are represented as well those in classes E_3, E_5. (Cf. Figure 3.3.) As κ increases to the critical level $\kappa = 2$ the area of the curve representing $\widehat{E}_5 = L - E_{45}$ shrinks to zero. Hence at this critical level the regular point z_1 of intersection of $C_{L-E_{45}}$ with $D_2 := \mathbb{CP}_{12}^1$ "cancels" the singular point z_2 on D_1. Thus, going back to the manifold M_2, there are isotropy spheres of order 2 between the critical points x_4 and x_2 at levels -6 and 2 respectively and between the points x_3 and x_1 at levels -2 and 6 respectively. This should be contrasted with the situation when $\ell = 4$ or 5; cf. Figure 3.4.

When $\ell = 3$ the analogous formulas are

$$\widehat{L} = 4L - 2E_{123} - E_{456}, \quad \varepsilon_6 = 3L - E_{123} - 2E_{456},$$

$$\widehat{E}_i = L - E_{jk}, \text{ for } \{i, j, k\} = \{1, 2, 3\}, \quad \widehat{E}_i = 2L - \sum_{j \neq i} E_j \text{ for } i = 4, 5, 6,$$

$$\Phi_J^*([\omega_\kappa]) = (1 + \tfrac{\kappa}{6})\big(3a - e_1 - e_2 - e_3\big) - (1 + \tfrac{\kappa}{2})\big(e_4 + e_5 + e_6\big)$$

There is no toric model in this case because $(Z, \omega_\kappa), |\kappa| < \ell$, is constructed by embedding the ellipsoid $\lambda E(1, \ell)$ into $E(2, 3)$ for $\lambda < \frac{12}{6+\ell}$ and when $\lambda > 3/\ell$ there is no linear embedding that does this. However, because $M_3 = \widetilde{G}_{\mathbb{R}}(2, 5)$ supports a Hamiltonian T^2-action, there should be an S^1-equivariant embedding $\lambda E(1, 3) \rightarrow E(2, 3)$ for all
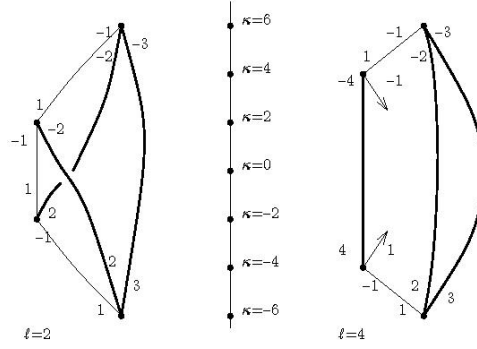
FIGURE 3.4. Diagrams of the S^1 invariant gradient spheres (i.e. gradient flow lines of H with respect to an invariant metric) when $\ell = 2, 4$. The isotropy spheres are marked with thicker lines. When $\ell = 2$ the diagram is a projection of the edges of a 3-simplex because one can assume that the metric is invariant under a R^3 action.

$\lambda < \frac{4}{3}$. Note that as in the case $\ell = 2$ there is an isotropy sphere (of order 3 this time) between the points x_1 and x_3 and between x_2 and x_4. This was noticed by Tolman in [29]; one can check it by calculating the Chern class of the isotropy spheres. (Recall that the Chern class of the S^1 orbit of a gradient flow line from q to q' is the difference in the sum of the isotropy weights at q, q'; cf. [20] for example.)

4. COMPLEX STRUCTURES ON M_ℓ.

Suppose that J is a $\mathbb{C}^*$ invariant complex structure on a complex manifold M, choose a Kähler metric on M that is invariant under the associated S^1 action, and consider the corresponding Hamiltonian function H. Then the reduced space at a regular level κ of H can be identified with the quotient $U_\kappa / \mathbb{C}^*$ where U_κ is the union of all $\mathbb{C}^*$ orbits that intersect the level set $H^{-1}(\kappa)$. Since U_κ changes only when κ passes a critical value, the induced complex structure on the reduced spaces is constant in each interval I of regular values. Next observe that the regular levels $H^{-1}(\kappa), \kappa \in I$, fit together to form a subset $\mathcal{S}$ of a holomorphic line orbibundle $\mathcal{L} \to Z$ whose fibers over the points of Z are (varying) annuli. (This holds because the fibers of the (holomorphic) projection $H^{-1}(I) \to Z$ support an S^1 action that extends to a local $\mathbb{C}^*$ action.) Thus M can be considered as a completion of $\mathcal{L} \smallsetminus \mathcal{L}_0$, where $\mathcal{L}_0$ denotes the zero section of $\mathcal{L}$. Therefore, we will approach the construction of a $\mathbb{C}^*$-invariant complex structure on M_ℓ by first finding a suitable complex structure J on Z and then a suitable orbibundle $\mathcal{L}$. As usual, we construct J on Z by finding a suitable complex structure on the resolution X_k.

To do this, it is convenient to change our point of view, thinking of the gluing map ψ of diagram (3.2) as being the identity map, and the induced map $L, E_i \mapsto \widehat{L}, \widehat{E}_i$ (or $\widetilde{L}, \widetilde{E}_i$) as corresponding to a different choice of basis for $H_2(X_k)$. Moreover the fact that ψ reverses the Euler class χ_k translates into the fact that the formula expressing

the class $\varepsilon_k = PD(\chi_7)$ in terms of the first basis should be equal, apart from a sign change, to that expressing it in terms of the second basis.

For clarity we shall now denote the second set of classes $\widehat{L}, \widehat{E}_i$ or $\widetilde{L}, \widetilde{E}_i$ by L', E'_i. We show below (in Lemma 4.1) that it is possible to choose the homology classes L', E'_i so that the set

$$\mathcal{H}'_0 := \{L' - E'_{123}, E'_i - E'_{i+1}, i \neq 3, k\} \subset H_2(Z; \mathbb{Z})$$

coincides with

$$\mathcal{H}_0 = \{L - E_{123}, E_i - E_{i+1}, i \neq 3, k\}.$$

Therefore, if these classes have smooth J-holomorphic representatives, we can think of the complex space (Z, J) as obtained either by contracting the curves in $\mathcal{H}_0$ or those in $\mathcal{H}'_0$. If the complex structure J on X_7 also has the property that the classes E_3, E_k and E'_3, E'_k have smooth representatives, then we can identify (Z, J) with the weighted blow up $\mathbb{C}P^2_{1,2,3} \# \overline{\mathbb{C}P}^2_{1,1,\ell}$ in two ways, identifying the divisors $D_1 = \mathbb{C}P^1_{2,3}$ and $D_2 = \mathbb{C}P^1(1, \ell)$ either with the images of E_3 and E_k or with the images of E'_3 and E'_k. We show in Proposition 4.5 below how these ideas lead to a construction of M_ℓ as a complex manifold. When there is no danger of confusion we shall sometimes write E_i for the (unique) J-holomorphic representative in class E_i.

The first step is to find suitable homology classes for L', E'_i. We shall do this first for the case $k = 7$. In this case, define

$$(4.1) \quad \begin{array}{rcl} E'_i : & = & 2L - E_{j4567}, \text{ where } (i, j) = (1, 3), (2, 2), (3, 1), \\ E'_i : & = & 3L - 2E_j - \sum_{m \neq j} E_m, \text{ where } (i, j) = (4, 7), (5, 6), (6, 5), (7, 4) \\ L' : & = & 7L - 2E_{123} - 3E_{4567}, \\ \varepsilon'_7 : & = & \frac{1}{12}(6L' - 2E'_{123} - 3E'_{4567}) \end{array}$$

The proof of the next lemma is left to the reader. Note that the somewhat complicated labelling of the E'_i was chosen so that each $E'_i - E'_{i+1}$ in $\mathcal{H}'_0$ equals some $E_j - E_{j+1}$ in $\mathcal{H}_0$.

Lemma 4.1. (i) *There is an automorphism α of $H_2(X_7)$ that takes L, E_i to L', E'_i respectively. Moreover $\varepsilon'_7 = \alpha(\varepsilon_7) = -\varepsilon_7$.*
(ii) $\mathcal{H}_0 = \mathcal{H}'_0$.

Lemma 4.2. (i) *There is a complex structure J on X_7 for which the classes L, E_3, E_7, E'_3, E'_7 as well as those in $\mathcal{H}_0$ have smooth holomorphic representatives.*
(ii) *This J is unique up to biholomorphism.*

Proof. If J is a complex structure satisfying the hypotheses of (i), then we may successively blow down E_3, E_2, E_1 to a point p and also E_7, E_6, E_5, E_4 to a point q. The blow down manifold is diffeomorphic to $\mathbb{C}P^2$ with its unique complex structure. This blow down map takes $L - E_{123} \in \mathcal{H}_0$ to a line through p that we shall call R. Further, it takes the embedded curve E'_7 to an immersed cubic T with a node at q that is triply tangent to R at p. Thus p is a flex point on T. Further the curve in class E'_3 is taken to a conic Q through p that has a four-fold tangency to T at q.

We claim that, up to projective transformation, there is at most one configuration of this kind. To see this, note that given T and a choice of flex point p, the conic Q is determined by the further choice of a branch B of T at its unique node q. But there is a unique choice of T, p, q, B up to projective transformation. In fact, because all nodal cubics are projectively equivalent, we may suppose that T is given by the equation $F = 0$ where $F = z_3(z_1^2 - z_2^2) - z_1^3$ with node at $q = [0 : 0 : 1]$. Its three flex points are the points on $T \smallsetminus \{q\}$ where the Hessian $\left| \frac{\partial^2 F}{\partial z_i \partial z_j} \right|$ vanishes, and one can check directly that these are permuted transitively by projective transformations that preserve T. Therefore we may take $p = [0 : 1 : 0]$ (where T is triply tangent to the line at infinity $z_3 = 0$). Note finally the reflection $[z_1 : z_2 : z_3] \mapsto [-z_1 : z_2 : z_3]$ interchanges the two branches at $q = [0 : 0 : 1]$.

Since there is a unique blowing up process that converts T and Q to curves in X_7 in classes E_7', E_3', this proves (ii). To prove (i) it remains to check that there is a configuration of curves T, Q with the required properties. But given T as above, let Q be the unique conic that intersects the branch B to order 4 at q and also intersects T at p. To see such Q exists, consider the family of conics through the points $p, q = x_4, x_5, x_6, x_7$ where $x_i \in T$, and let the three points x_5, x_6, x_7, converge along the branch B to q. Then the limiting degree 2 curve intersects T at q to order 5 and so cannot degenerate into a pair of lines. (The two lines would have to consist of tangents to T at the node q, but these are triple tangents and so do not also go through $p \in T$.) $\square$

When $\ell = 5$ we argue similarly, using the formulas:

$$(4.2) \qquad L' := 16L - 5E_{123} - 6E_{4\ldots8}, \quad \varepsilon_8' = \tfrac{1}{30}(15L' - 5E_{123}' - 6E_{4\ldots8}')$$

$$E_i' := 5L - E_{jk} - 2E_{i4\ldots8}, \quad \text{for } \{i, j, k\} = \{1, 2, 3\},$$

$$E_i' := 6L - 3E_j - 2 \sum_{m \neq j} E_m, \quad \text{for } i, j \in \{4, \ldots, 8 \,|\, i + j = 12\}$$

$$\mathcal{H}_0' := \{L' - E_{123}', E_i' - E_{i+1}', i \neq 3, 8\}$$

It is easy to check the analog of Lemma 4.1, while Lemma 4.3 below replaces Lemma 4.2.

Lemma 4.3. (i) *There is a complex structure J on X_8 for which the classes L, E_3, E_8, E_3', E_8' as well as those in $\mathcal{H}_0$ have smooth holomorphic representatives.*

(ii) *Moreover J is unique up to the choice of a rational parameter $\mu \in \mathbb{C}P^1 \smallsetminus \mathcal{F}$, where $\mathcal{F}$ is a finite set.*

Proof. Fix points $p \neq q$ in $\mathbb{C}P^2$, a line R through p but not q and a conic Q through q and not p. We shall assume that the tangent line to Q from p does not go through q. In the following construction we assume that p and R are fixed but allow q to vary on Q. We shall construct $J = J_q$ on X_8 by blowing up p three times and q five times. The blow ups at p are directed by the line R as in the construction of X_k after Lemma 2.6. Similarly, the five fold blow up at q is directed by Q; thus the classes $E_4 - E_5, \ldots, E_7 - E_8$ and $2L - E_{45678}$ are all represented by smooth curves.

Step 1: *For generic q, the class $3L - 2E_4 - \sum_{m=1, m \neq 4}^{8} E_m$ is not represented in (X_8, J).*

Let T_q be a nodal cubic that is triply tangent to R at p and has node at q with one branch B_q at q tangent to Q to order 4. Such a curve exists by the proof of Lemma 4.2, and is unique because its proper transform T_q' under the first 7 blow ups is an exceptional sphere in the class $3L - 2E_4 - \sum_{m=1,m\neq 4}^{7} E_m \in \mathcal{E}(X_7)$. If Q' is the proper transform of Q under these blowups then

$$T_q' \cdot Q' = \left(3L - 2E_4 - \sum_{m=1,m\neq 4}^{7} E_m\right) \cdot \left(2L - E_{4567}\right) = 1.$$

The class $3L - 2E_4 - \sum_{m=1,m\neq 4}^{8} E_m$ is represented in (X_8, J) exactly if the point of intersection $T_q' \cap Q'$ blows down to q, that is, exactly if the branch B_q is tangent to Q to order 5 at q. We claim that this does not happen for generic q. Because the set $\mathcal{F}_Q$ of $q \in Q$ for which this happens is algebraic and Q has dimension 1, it suffices to show that $\mathcal{F}_Q \neq Q$.

Suppose that $q \in \mathcal{F}_Q$. Let $Q'' \neq Q$ be a conic that is tangent to Q to order 4 at q. Then B_q is not tangent to Q'' to order 5. Moreover, there is a projective transformation Φ of $\mathbb{C}P^2$ that fixes p, R and takes Q to Q''. Let $q_0 := \Phi^{-1}(q) \in Q$. Then the unique nodal cubic T_{q_0} must coincide with $\Phi^{-1}(T_q)$. Moreover, T_{q_0} is not tangent to Q at q_0 to order 5 because $\Phi(T_{q_0}) = T_q$ is not tangent to $\Phi(Q) = Q''$ at $\Phi(q_0) = q$ to order 5 by construction. Hence $q_0 \in Q \setminus \mathcal{F}_Q$.

Step 2: *For generic q the classes E_3' and E_8' have smooth holomorphic representatives.*

Since $E_3', E_8' \in \mathcal{E}(X_8)$ they have nontrivial Gromov–Witten invariants and hence have holomorphic representatives for all q. Therefore we just need to check that these representatives are irreducible. We will consider representatives S' for E_8'; the argument for E_3' is similar.

If S' were not smooth it would be the union of components S_i' in classes either of the form $d_i L - \sum_{k=1}^{8} m_{ik} E_k$ with $d_i > 0$ and $m_{ik} \geq 0$ or with $d_i = 0$ and in the set $\mathcal{H}_0 \cup \{E_3, E_8\}$. Since these classes sum to E_8', we must have $\sum d_i = 6$. Moreover, because the curves in $\mathcal{H}_0$ are represented, as is the class $2L - E_{4\ldots8}$ (the proper transform of Q), any component that does not lie in $\mathcal{H}_0$ must satisfy the conditions

(4.3)
$$d_i \geq m_{i1} + m_{i2} + m_{i3}, \quad 2d_i \geq \sum_{i=4}^{8} m_{ik},$$

$$m_{i1} \geq m_{i2} \geq m_{i3}, \quad m_{i4} \geq m_{i5} \geq \cdots \geq m_{i8},$$

$$\sum_k (m_{ik}^2 - m_{ik}) \leq 2 + d_i^2 - 3d_i.$$

The first conditions above come from positivity of intersections, while the last comes from the fact that these curves are rational and so must satisfy the genus zero adjunction inequality $c_1(S_i) \leq 2 + (S_i)^2$. This means that if $d_i = 3$ at most one of the m_{ik} is > 1 and that all $m_{ik} \leq 2$. In other words, the m_{ik} (listed in decreasing order) are at most $(2, 1, \ldots, 1)$. Similarly if $d_i = 4$ the m_{ik} are at most $(2, 2, 2, 1, \ldots, 1)$ or $(3, 1, \ldots, 1)$, while if $d_i = 5$ they are at most $(3, 3, 1, \ldots, 1)$ or $(3, 2, 2, 2, 1, \ldots, 1)$ or $(2, \ldots, 2, 1, 1)$.

Thus, if $d_i = 3$ the only permissible class with all $m_{ik} \neq 0$ is $(3; 1, 1, 1, 2, 1, \ldots, 1)$, where here we have listed the m_{ik} in order of increasing k. But, by Step 2, this element is not represented. (If it were represented we could decompose E_8' as twice this class plus $(E_4 - E_5) + \cdots + (E_7 - E_8) + E_8$.)

Similarly, E_8' would decompose if either of the lines $L - E_{145}$ or $L - E_{1234}$ were represented, since E_8' is the sum of $6(L - E_{145})$ or $2(L - E_{1234}) + 2(2L - E_{4\ldots8})$ with suitable classes from $\mathcal{H}_0 \cup \{E_3, E_8\}$. However, for generic q these classes are not represented either. The reader can now check that there are no permissible decomposition of E_8'. For example, if one of the curves is in class $(4; 2, 2, 2, 1, 1, 1, 1, 1)$ one could add $(2; 0, 0, 0, 1, 1, 1, 1, 1)$, but this does not give a large enough coefficient for E_4. Also, because of the conditions $m_{i1} \geq m_{i2} \geq m_{i3}$ and $m_{i4} \geq \cdots \geq m_{i8}$ it does not help to consider classes with $m_{i1} > m_{i3} = 0$ or $m_{i5} > m_{i8} = 0$ since there would have to be other elements in the decomposition with $m_{j3}, m_{j8} \neq 0$, which would make $\sum d_i$ too large.

These two steps complete the proof of (i).

Step 3: *Proof of* (ii).

Suppose that J' is any complex structure on X_8 for which the classes E_3' and E_8' as well as those in $\mathcal{H}_0$ have smooth holomorphic representatives. Then the classes $L - E_{1234}$, $L - E_{145}$ and $3L - 2E_4 - \sum_{m=1, m \neq 4}^{8} E_m$ cannot have holomorphic representatives since they have negative intersection with E_8'. Also, any class that is represented by a rational curve must satisfy all the conditions in (4.3) except for $2d_i \geq \sum_{k \geq 4} m_{ik}$ by positivity of intersection with $\mathcal{H}_0$. Hence the class $2L - E_{4\ldots8} \in \mathcal{E}(X_8)$ has a (unique) embedded representative because none of its decompositions satisfy these conditions.

Since the classes in $\mathcal{H}_0$ are represented, there is a blow down map $\pi : (X_8, J') \to \mathbb{C}P^2$ that collapses the curves $E_k - E_{k+1}$ for $k \neq 3, 8$. Let p be the image of $E_1 - E_2$ and q the image of $E_4 - E_5$. We define the conic Q to be the blow down of the curve in class $2L - E_{4\ldots8}$. Next observe that all triples (p, R, Q) consisting of a conic Q, a point $p \notin Q$ and a line R through p are projectively equivalent, provided that R is not tangent to Q. Moreover the only way to blow up $\mathbb{C}P^2$ to a complex structure on X_8 for which the curves in $\mathcal{H}_0$ as well as $2L - E_{4\ldots8}$ are represented is to perform repeated blow ups directed by R at p and by Q at some point $q \in Q$ as described at the beginning of the proof. Hence the only choice in the above construction is the rational parameter q. $\square$

We shall denote by J the complex structure on Z induced by the blow down map $(X_k, J) \to Z$ where J is as constructed in the previous two lemmas. As explained at the beginning of §2, it is also possible to construct Z as a toric manifold, with moment polytope as in Figure 2.2. Let us denote the corresponding complex structures on Z and X_k by J_T. As we pointed out in Remark 2.12, J_T is not equal to the complex structure J in Lemmas 4.2 and 4.3 since in the toric case the blowups at q are also directed by a line (rather than by the conic Q) — lines can be chosen to be invariant under the group action while conics cannot be. Thus the class $L - E_{4\ldots k}$ always has a J_T holomorphic representative.

Both complex structures J and J_T on Z are obtained by blowing up the weighted projective space $\mathbb{C}P^2_{123}$ at a point q not on the exceptional divisor D_1. Therefore the divisor D_2 is represented in both cases. (In fact, D_1, D_2 are the images of E_3 and E_8 respectively.) Moreover, the complex structures J and J_T coincide on $Z \smallsetminus D_2$.

Lemma 4.4. *For any open neighborhood U of D_2 in Z there is a closed neighborhood $V \subset U$ of D_2 in Z, and a diffeomorphism $f : (Z, J_T) \to (Z, J)$ that is the identity in $Z \smallsetminus U$ and on D_2 and is a biholomorphism near V.*

Proof. Because D_2 is resolved in X_k by a negative divisor D'_2 (consisting of the curves in classes $E_m - E_{m+1}$ for $4 \leq m < k$ and E_k), there is a unique complex structure near D'_2 in X_k and hence a unique structure near D_2 in Z. Therefore the identity map on D_2 extends to a diffeomorphism $g : (V, J_T) \to (Z, J)$ on some neighborhood V of D_2 that is a biholomorphism onto its image. Since $g = id$ on D_2, it is easy to find a diffeomorphism of Z that equals g on some shrinking of V and the identity outside U. $\square$

We are now in a position to prove the second statement in Theorem 1.1.

Proposition 4.5. *M_ℓ has a $\mathbb{C}^*$-invariant complex structure when $\ell = 4, 5$. This is unique up to $\mathbb{C}^*$-equivariant biholomorphism when $\ell = 4$, and depends on a rational parameter when $\ell = 5$.*

Proof. First consider the case $\ell = 4$. We will construct M_4 to be a holomorphic manifold with a holomorphic S^1 action. Then it will automatically have a $\mathbb{C}^*$ action. M_4 will be the union of 3 pieces corresponding to the three intervals $(-6, -3), (-4+\varepsilon, 4-\varepsilon), (3, 6)$ of values of the moment map. We construct the middle slice first.

Denote by $\pi : X_7 \to Z$ the map obtained by collapsing the curves in $\mathcal{H}_0$. Let J be the complex structure on X_7 constructed in Lemma 4.2 and denote also by J the induced complex structure $\pi_*(J)$ on Z. Since holomorphic line bundles are determined by elements of $H^1(\cdot, \mathbb{C}^*)$, it follows from Lemma 2.9 that there is a unique line bundle $\mathcal{L} \to (X_7, J)$ with Euler class ε_7, and that this bundle descends to a holomorphic orbibundle $\overline{\mathcal{L}}$ over (Z, J). Note that the total space of $\overline{\mathcal{L}}$ is smooth because the boundary of the neighborhood $\mathcal{V}$ of the curves in $\mathcal{H}_0$ is smooth.

Take any Hermitian metric on $\overline{\mathcal{L}}$, pull it back to $\mathcal{L}$ and then, given real valued functions $0 < R_1 < R_2$ on Z define the slices $\mathcal{S}_0, \overline{\mathcal{S}}_0$ by setting

$$\mathcal{S}_R = \{(x, v) \in \mathcal{L} : R_1(\pi(x)) < |v| < R_2(\pi(x))\},$$
$$\overline{\mathcal{S}}_R = \{(z, v) \in \overline{\mathcal{L}} : R_1(z) < |v| < R_2(z)\}.$$

The manifold $\overline{\mathcal{S}}_R$ (for suitably small R_1 and large R_2) is the middle part of (M_4, J). Note that it has an S^1 action obtained by multiplication by $e^{i\theta}$ in the fibers of $\mathcal{L}$.

We need to complete $\overline{\mathcal{S}}_R$ at both its ends by attaching holomorphic manifolds that are diffeomorphic to $M^{<-3}$ and $M^{>3}$. Let us first consider how to attach the lower half $M^{<-3}$. If we think in terms of the $\mathbb{C}^*$ orbits i.e. the fibers of $\mathcal{L}$ (forgetting the moment map), we need to compactify $\mathcal{L} \smallsetminus \mathcal{L}_0$, replacing the zero section $\mathcal{L}_0$ by a copy of $\mathbb{C}^* \cup \{0\} \cup \{\infty\}$. Here we can think of $\{0, \infty\}$ as the fixed points at levels $\kappa = -6, -4$

respectively. It is hard to see how to construct such a compactification from $\mathcal{L}$ itself. In particular, when passing $\kappa = -4$ we need to collapse the divisor D_2 in the zero section $\mathcal{L}_0$ to a point and also begin a new $\mathbb{C}^*$ orbit.

However there is a toric model for this process: the discussion before Lemma 2.11 implies that the set $M^{<-3}$ (considered as a smooth manifold) can be constructed with a toric structure and so has a corresponding complex structure J_T. Therefore it suffices to modify J_T so that it matches the complex structure J that we already have on $\overline{\mathcal{S}}_R$.

As above, the slice $(M^{(-4,-3)}, J_T) := \left(H^{-1}\big((-4,-3)\big), J_T\right)$ can be considered as a subset of a holomorphic orbibundle $\pi : (\mathcal{L}_T, J_T) \to (Z, J_T)$ with Euler class that pulls back to ε_7. Since the Euler class uniquely determines the bundle, it suffices to change J_T to $f^*(J)$ in the open set $U \smallsetminus V$ and make the corresponding modification to the complex structure of $\mathcal{L}_T$, where the notation is as in Lemma 4.4. This defines a new complex structure in $M^{(-4,-3)}$ that we shall call J. To see that this extends over the whole of $M^{<-3}$, observe that we have not changed the structure near $\pi^{-1}(D_2)$ so that J extends over the critical level $\kappa = -4$ to the sets $M^{(-6+\delta,-3)}$ for all $\delta > 0$. But the slices $(M^{(-6+\delta,-4)}, J)$ are subsets of a holomorphic orbibundle over $(\mathbb{C}P^2_{123}, J)$, and J is diffeomorphic to J_T on $\mathbb{C}P^2_{123}$. (This follows by construction.) Hence we may identify $(M^{(-6+\delta,-4)}, J)$ with an appropriate subset of the canonical bundle over $(M^{(-6+\delta,-4)}, J_T)$ and therefore extend J over the critical level $\kappa = -6$ by the toric structure. Note that the resulting complex structure J on $M^{<-3}$ admits a holomorphic S^1 action given by multiplying by $e^{i\theta}$ in the fibers of the bundle.

Now observe that if we choose R_1 suitably we can compactify this end of $\overline{\mathcal{S}}_R$ by attaching $(M^{<-3}, J)$. Again, the union $\overline{\mathcal{S}}_R \cup M^{<-3}$ has an S^1 action. Similar remarks apply to the other end. In fact the involution $(x, v) \mapsto (x, 1/\overline{v})$ (where $v \mapsto \overline{v}$ is complex conjugation) takes $\mathcal{L} \to X_7$ to $\mathcal{L}^* \to X_7$. Since $e(\mathcal{L}^*) = -\chi_7 = \chi_7'$ we can repeat the above argument replacing the classes $\mathcal{L}, E_i$ by $\mathcal{L}', E_i'$.

This constructs the $\mathbb{C}^*$-invariant complex structure on M_4. It is unique up to $\mathbb{C}^*$ equivariant biholomorphism because the complex structure on the reduced space is unique and there are no other choices in the construction. This completes the proof in the case $\ell = 4$. The case $\ell = 5$ is almost identical, and is left to the reader. $\square$

Remark 4.6. Since the fixed point data of the S^1 actions considered here are symmetric under the inversion $S^1 \to S^1 : \theta \mapsto -\theta$, the uniqueness result implies that there is an S^1-equivariant symplectomorphism of (M_ℓ, Ω) that reverses the S^1 action. Further, because there is a unique S^1 invariant complex structure when $\ell = 4$, this map can be taken to be a biholomorphism in this case. (In fact the existence of such action reversing maps is obvious from our construction.) However, when $\ell = 5$ and $J_\lambda, \lambda \in S^2 \smallsetminus \{\text{finite set}\}$, is a generic S^1 invariant complex structure on M_5 then we cannot expect there to be a corresponding biholomorphic map; rather there should be a holomorphic involution τ of the parameter space such that reversing the S^1 action takes (M, J_μ) to $(M, J_{\tau(\mu)})$. In the case that the holomorphic S^1 action (M, ω) extends to a holomorphic action of $SO(3)$, then because the S^1 action is conjugate to its inverse by an element in $SO(3)$, the corresponding parameter μ is fixed by τ. Since the

Mukai–Umemura 3-fold does have a holomorphic $SO(3)$ action, it might be interesting to look at it from this point of view. One could also try to to analyze it using methods to study Hamiltonian $SO(3)$ actions such as those developed by R. Chang [1].

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